Loop quantum gravity: status and some results

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• A brief introduction to loop quantum gravity.
• Some of its results.
• Contentious issues.
• Recent progress in mini and midi-superspace models.
Micro-Introduction to loop quantum gravity:

A bit of history:

In the mid 1980’s Abhay Ashtekar showed that one could rewrite the equations of general relativity in terms of a set of canonical variables that looked like those of an SU(2) Yang-Mills theory $\tilde{E}_i^a, A^i_a$.

This raised hopes that one could import many techniques used for the quantization of Yang-Mills theory to the gravitational context.

This hope was achieved but perhaps in a different way that it was originally thought.

Some techniques from Yang-Mills theories were indeed imported. For example, the use of holonomies (loops). Rovelli and Smolin showed that one could construct a quantum representation of gravity based on loops with some similarities to the one Gambini and Trias had constructed for Yang-Mills theory.

However, gravity and Yang-Mills are profoundly different theories. Among the key differences are the symmetries. Gravity is invariant under diffeomorphisms.

The description in terms of loops appeared naturally suited to deal with diffeomorphism invariance. Yet it required the use of novel mathematical techniques to actually implement this.
Ashtekar, Lewandowski and others developed in the mid 90’s the techniques needed to construct a loop representation for a diffeomorphism invariant theory like gravity.

Among the techniques was the introduction of a well defined diffeomorphism invariant measure in the space of connections modulo gauge transformations. The resulting space of wavefunctions is radically different from that in Yang-Mills theories.

**Some results:**

Using this space of wavefunctions in the second half of the 90’s Thiemann was able to construct a mathematically consistent theory of quantum gravity, representing the Hamiltonian constraint (Wheeler-DeWitt) as a well defined quantum operator.

It remains to be seen if this theory of quantum gravity reproduces nature correctly. In particular, it is difficult to extract semiclassical information from the diffeomorphism invariant quantum context.
The theory has a beautiful kinematical structure, with areas and volumes having discrete spectra. This allows to compute the degrees of freedom associated with the horizon of a black hole and show that the entropy is proportional to the area.

In a nutshell, the construction of a mathematically well defined theory of quantum gravity and the computation of the black hole entropy are perhaps the most salient results obtained by the whole approach.

In addition to these developments related to canonical quantization, there is a significant group of people applying similar techniques to the covariant construction of the path integral ("spin foams"). Some level of success constructing a propagator that yields Newton’s law has been achieved recently by the group in Marseille (Rovelli and collaborators). I will concentrate in the canonical approach in this talk.
The Hilbert space that loop quantum gravity uses is rather different than Hilbert spaces commonly used in other field theories. In particular it is not weakly continuous: expectation values of operators depending on a parameter do not change continuously when one changes the parameter value. Therefore the Hilbert space does not admit a countable basis (it is non-separable).

The loop states (“spin network states”) $|\Gamma\rangle$ where $\Gamma$ is a graph in space have an inner product $<\Gamma|\Gamma'>$ that is zero if $\Gamma$ and $\Gamma'$ are not diffeomorphic to each other. This has been compared to the introduction of the discrete topology of the real line with countable unions of points as open sets. The only notion of “closeness” between points is if they coincide and therefore any function is continuous in this topology.

These properties appear rather counterintuitive. On the other hand, one expects that counterintuitive elements may have to be introduced to overcome the issues facing conventional quantum gravity.

Because the space of states is so different from the usual Fock space, it is challenging to see how semi-classical coherent states emerge in this picture. This is further compounded by the issue that in gravity one has to choose appropriate observables (in a diffeo invariant context) to set up questions.

In simple examples, like the harmonic oscillator, it has been shown that these types of quantizations admit states (complicated superpositions) that approximate the usual Fock space coherent states.
Some of the contentious issues:

Loop quantum gravity has been criticized in various fora. Perhaps most remarkable are the papers by Nicolai, Peeters and Zamarklar, and shorter but more up to date, Nicolai and Peeters. These papers are carefully written and the criticisms well explained. Thiemann has responded in detail in a paper to the first article.

Some of the criticisms stem from skepticism that the Hilbert space mentioned in the previous slide can really yield the correct semiclassical theory. Even the example of the harmonic oscillator generated some debate in the literature with some critics not entirely satisfied (Policastro, Hellings).

Another salient criticism is that the viewpoint is that space-time will in the end have some level of discreteness. This point of view is quite different from, say, lattice Yang-Mills theory. There discreteness is introduced as a computational device, but one finally takes a continuum limit.

The possible lack of a continuum limit raises the possibility that the theory will suffer ambiguities, and that these will be severe. It also raises the possibility that Lorentz invariance will be unacceptably violated.
Another issue is the general covariance of the approach. In a canonical theory, invariance under diffeomorphisms is broken by the 3+1 split. General covariance is assured by the presence of a constraint algebra. Classically, one has,

\[ \{ D, D \} \approx D \]
\[ \{ D, H \} \approx H \]
\[ \{ H, H \} \approx qD \]

Because the Hilbert space is not weakly continuous, one cannot define a quantum operator corresponding to D. Therefore one cannot check the above algebra. One can consider diffeomorphism invariant states and check that \{H,H\}~0. (The story is more complicated technically since H is not diffeo invariant).

An issue that critics raise is how could loop quantum gravity recover general covariance when it treats diffeomorphisms and Hamiltonian in such different ways?
The difficulties arising from the constraint algebra have led to new proposals for the quantization of constrained systems that are more general than the Dirac method. For example, Thiemann’s “master constraint programme” or our proposal with Gambini called “uniform discretizations”. Both proposals, broadly speaking, seek to handle systems with constraints that cannot be imposed entirely at a quantum level and are being tested with various examples of increasing complexity.

Working with mini/midi superspaces:

To try to settle many of the raised issues in the full theory is close to impossible. Not only are the issues subtle, but the dynamics of full GR is expected to be hard. This has led several of us to try to probe things in simpler scenarios, where one has more control.

It is interesting to notice that the use of the non-standard Hilbert space structures we mentioned before already has consequences even for finite dimensional models.

Roughly speaking, it is equivalent to pursue a quantum mechanics where the operator $p$ is well defined but $x$ is not. Instead, what is well defined is $\exp(ikx)$ and the limit $k \to 0$ does not exist.

Such quantizations have already been pursued for cosmological models (“loop quantum cosmology” pursued by Bojowald, Ashtekar and others).
When one works in mini-superspaces the issue of the algebra of constraints does not arise (there is only one constraint). When one goes to midi-superspaces (e.g. spherical Symmetry) the problem arises again. One way to bypass the problem is to further gauge fix in such a way that the algebra of constraint simplifies. This can be done in the spherically symmetric case.

In the next slides I would like to show some results for spherical symmetry.
Spherical symmetry with the new variables

Previous work with the new variables, Bengtsson (1988) Kastrup and Thiemann (1993) and Bojowald and Swiderski (2005, 2006). Choose connections and triads adapted to spherical symmetry,

\[
A = A_x(x) \Lambda_3 dr + (A_1(x) \Lambda_1 + A_2(x) \Lambda_2) d\theta + ((A_1(x) \Lambda_2 - A_2(x) \Lambda_1) \sin \theta + \Lambda_3 \cos \theta) d\varphi,
\]

\[
E = E^x(x) \Lambda_3 \sin \theta \frac{\partial}{\partial x} + (E^1(x) \Lambda_1 + E^2(x) \Lambda_2) \sin \theta \frac{\partial}{\partial \theta} + (E^1(x) \Lambda_2 - E^2(x) \Lambda_1) \frac{\partial}{\partial \varphi},
\]

\(\Lambda\)'s are generators of \(\text{su}(2)\).

It is convenient to make some changes of variables, in particular to go to variables where the Gauss law is already solved. We will omit details of these changes here.
The final choice yields two conjugate pairs $E^x, K_x$ and $E^\phi, K_\phi$

Their relation to the traditional metric variables is,

$$g_{xx} = \frac{(E^\phi)^2}{|E^x|}, \quad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\text{sign}(E^x) \frac{(E^\phi)^2}{\sqrt{|E^x|}} K_x, \quad K_{\theta\theta} = -\sqrt{|E^x|} \frac{A_\phi}{2\gamma},$$

The constraints take a relatively simple form with the usual 1+1 diffeo/Hamiltonian algebra of constraints (with structure functions),

$$D = -(E_x)' K_x + E^\phi (K_\phi)'$$

$$H = -\frac{1}{2} \frac{E^\phi}{\sqrt{E^x}} - 2K_\phi \sqrt{E^x} K_x - \frac{1}{2} \frac{E^\phi K_\phi}{\sqrt{E^x}} \frac{E^x}{\gamma} + \frac{1}{8} \frac{((E^x)')^2}{\sqrt{E^x} E^\phi} - \frac{1}{2} \frac{\sqrt{E^x} (E^x)' (E^\phi)'}{(E^\phi)^2} + \frac{1}{2} \frac{\sqrt{E^x} (E^x)''}{E^\phi},$$

The quantization of this model directly is therefore a hard thing since it has the “problem of dynamics”. One could use the master constraint or the consistent discretizations. We will further gauge fix to avoid this problem.
We start by fixing a gauge $E^x = f(x,t)$. This determines the Lagrange multiplier (shift) $N^r = \dot{f}(x,t)/f'(x,t)$ The variable $K_x$ is eliminated imposing the diffeomorphism constraint strongly.

One is left with $E^\phi$ and $K_\phi$ as canonical variables and with one (Abelian) constraint and a true Hamiltonian, since the gauge fixing breaks reparametrization invariance,

$$H_{true} = \int dx \frac{\dot{f}(x,t)}{f'(x,t)} E^\phi (K_\phi)' \quad \quad H_T = \int dx N^t \Phi + H_{True} + H_{Boundary},$$

$$\Phi = -\sqrt{E^x} - K^2 \sqrt{E^x} + \frac{1}{4} \frac{(E^x)'^2}{(E^\phi)^2} \sqrt{E^x} + 2M$$

For reasons of time we omit discussion of the boundaries, where treatment is straightforward.
Loop representation for the spherically symmetric case:

Manifold is a line. “Graph” is a set of edges $g = \bigcup_i e_i$.

To avoid presenting too many equations, I will write the states for the “gauge fixed” case we introduced. There the only variables in the bulk are $E^\varphi$ and $2\gamma K_\varphi = A_\varphi$

$$\mathcal{H} = L^2 \left( \otimes_N R_{\text{Bohr}}, \otimes_N d\mu_0 \right)$$

$$T_{g,\bar{\mu}}[K_\varphi] = \prod_{v \in V(g)} \exp \left( 2i \mu_v \gamma K_\varphi(v) \right)$$

$$\hat{E}_m^\varphi = -i\ell_{\text{Planck}}^2 \frac{\partial}{\partial K_{\varphi,m}}$$

$$\hat{E}_m^\varphi T_{g,\bar{\mu}} = \sum_{v \in V(g)} \mu_m \gamma \ell_{\text{Planck}}^2 \delta_{m,n(v)} T_{g,\bar{\mu}}$$

Volume of an interval $I$

$$V(I) = 4\pi \sum_{m \in I} |E_m^\varphi| \sqrt{|E^x|}$$

$$\hat{V}(I) T_{g,\bar{\rho}} = \sum_{v \in I} 4\pi |\rho_v| \sqrt{|E^x|} \ell_{\text{Planck}}^2 T_{g,\bar{\rho}}$$
Study of the classical holonomized theory:

Starting from the total Hamiltonian,

\[ H_T = - \int dx N' \left[ -\sqrt{E^x} - K^2 \sqrt{E^x} + \frac{1}{4} \left( \frac{(E^x)'^2}{(E^\varphi)^2} \right) + 2M \right] + \int dx N^r E^\varphi (K^\varphi)' , \]

We discretize it on a lattice as we described and holonomize

\[
H_{T}^{\mu} = - \sum_{0}^{L} (N_{n+1} - N_{n}) \left[ -\sqrt{E_{n}^x} - \frac{\sin(\mu K_{\varphi,n})^2}{\mu^2} \sqrt{E_{n}^x} + \frac{1}{4} \left( \frac{E_{n+1}^x - E_{n}^x}{(E_{n}^\varphi)^2} \right)^2 \sqrt{E_{n}^x} + 2M \right] \\
+ \sum_{0}^{L} N_{n} N_{n} E_{n}^\varphi \exp \left( i \frac{\mu}{\mu} (K_{\varphi,n+1} - K_{\varphi,n}) \right) - 1 
\]

To simplify, we will work in the continuum limit in which we make the separation of the points of the lattice vanish but keep \( \mu \) finite.

\[
H_{T}^{\mu} = - \int dx N' \left[ -\sqrt{E^x} - \frac{\sin(\mu K_{\varphi})^2}{\mu^2} \sqrt{E^x} + \frac{1}{4} \left( \frac{(E^x)'^2}{(E^\varphi)^2} \right) + 2M \right] + \int dx N^r E^\varphi (K^\varphi)' . 
\]

Recall that \( N^r \) is a given function, \( N^r = \dot{f}(x,t)/f'(x,t) \)
We will proceed to find a solution of the theory. That is, a solution of the constraint and of the evolution equations. A solution has to be found in a given gauge. So we will need to further gauge fix to have explicit results.

We would like to choose $f(x,t)$ in such a way that in the limit $\mu \to 0$ one recovers the usual Schwarzschild solution in Kruskal-like coordinates. That is, a metric with a singularity at $x^2-t^2=-1$. On the other hand, for finite $\mu$, we would like that surface to be a regular surface beyond which we can extend the metric.

We will choose $E^x=f_1(u,t,\delta)$ and $K_\varphi=f_2(u,t,\delta)$ with $u=x^2-t^2+1$ and $\delta$ a positive parameter such that when $\delta \to 0$, $\mu \to 0$ we get the usual Schwarzschild/Kruskal solution.
After quite a bit of trial and error, our (current) proposal for $E^x$ is,

$$E^x = \begin{cases} 
\left[ \frac{\delta (1 + u) + b^2 \left( 10u^2 + u^{7/2} \right) \left( \delta \left( t^2 - 1 \right) + 1 \right)}{u^{7/2} + (t^2 - 1) \left( \delta u^{7/2} + \delta^2 \right) + \frac{1}{2} \delta^2 u} \right] \\
\times \left[ \ln(1 + u)^2 + \delta^8 \right] \end{cases} M^2$$

With $b$ a function of $t$ that we will later see varies slowly. This choice has, for $u \to 0$, $E^x \to M^2 \delta^8$ independent of $t$, for large $u$ it goes as $b^2 \ln(u)^2 + \text{const.} + O(\ln(u)/u)$ and for $\delta \to 0$ it goes to zero for $u=0$ giving rise to the singularity.

For $K_\phi$ we chose,

$$K_\phi = -\frac{1}{2} \frac{\delta^{5/2}}{\mu} \frac{\pi \left( 1 + \ln \left( 1 + u^2 \right) \right)}{\left( \delta^{5/2} + \ln \left( 1 + u \right)^2 \right)} + \frac{t \ln \left( 1 + u^3 \right)}{u^{3/8}}$$

$$\times \left( -1 + \frac{u}{b(10 + u \ln(1 + u))} + \frac{(1 + 8b)u}{b^2 (100 + u \ln(1 + u)^2)} \right)$$

$$\times \left( \delta^2 t + \ln \left( 1 + u^3 \right) \right) \left( 1 + u^{1/8} \right).$$

This choice has, for $u \to 0$, that $K_\phi = \pi/(2 \mu)$ which implies $\sin(\mu \ K_\phi) \sim 1$ and the holonomized theory has maximal departure from usual GR at the place where one would have expected the singularity. In the limit $\delta \to 0$ it blows up at $u=0$ and one has the usual singularity. Its first derivative with respect to $x$ vanishes at $u=0$. 
We then solve the constraint for $E \phi$, and in order for it to be finite for $u=0$, one gets a relationship between $\mu$ and $\delta$.

$$\mu = \frac{\delta^2}{\sqrt{2 - \delta^4}}.$$

The derivative vanishes at $u=0$ as we wanted and the large $u$ behavior implies that

$$g_{xx}|_{u \to \infty} = 4 \frac{M^2 b^2}{u} + 8 \frac{M^2 b}{u \ln(u)},$$

Which is just $(1+2m/r)$ in these coordinates.

The evolution equations determine the lapse up to a quadrature, that can be done numerically.

$$N' = \frac{1}{4} \frac{\dot{K}_\phi (E^x)' - K'_\phi \dot{E}^x}{(1 - \frac{2M}{\sqrt{E^x}})^{3/2} \sqrt{E^x}}$$

This completes the determination of the solution, which we only have started to analyze. It is harder than it looks since the expressions are large and some are only known numerically.
To get a feel for what is going on, it is useful to plot $g_{xx}$ as a function of $r$. The red curve is the Kruskal-like solution we get in the limit $\delta \to 0$, the green curve the solution of the holonomized theory. This is for $\delta = 10^{-42}$ and for $t=100$. 
Global picture?

Main open questions: Uniqueness? Stability?
Summary:

• Loop quantum gravity provides a set of new ideas to quantize theories like general relativity.
• It has achieved some successes.
• There are issues that are contentious and unresolved that can be interpreted as “the glass half full” or “the glass half empty” at the moment.
• Exploring models of increasing complexity will ultimately shed light on which of the two points of view is the correct one.
• Even within the context of the simple models attractive physical results can emerge.
Some references:

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