Velocity-dependent Potential of a Rigid Body in a Rotating Frame

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We derive a velocity-dependent potential for describing the dynamics of a rigid body in a rotating frame. We show that, as for one-particle systems, the different components of this potential can be associated to electromagnetic analogs. Finally, we provide some simple examples to demonstrate the feasibility of using it as an alternative description of rigid body problems.

I. INTRODUCTION

In basic and advanced courses of Classical Dynamics, students are taught how to deal with a rotational frame of angular velocity $\vec{\omega}$ by including, besides the acting force $\mathbf{F}$, three inertial forces [1] onto Newton's equation, $m\ddot{\mathbf{r}} = \mathbf{F}$. For a particle of mass m and position $\mathbf{r} = \mathbf{r}(t)$, these are the Coriolis force,

$$\mathbf{F}_{\text{Coriolis}} = -2\, m\,\vec{\omega}\, \times \dot{\mathbf{r}}\,, \tag{1}$$

named after Gaspard-Gustave Coriolis (1792–1843) [2, 3] who described it in 1832–1835 (even though it was first introduced by Pierre-Simon Laplace (1749–1827) half a century before [4] in relation with tidal forces), the centrifugal force

$$\mathbf{F}_{\text{Centrifugal}} = -\, m\,\vec{\omega} \times (\vec{\omega} \times \mathbf{r})\,, \tag{2}$$

and the Euler force [5]

$$\mathbf{F}_{\text{Euler}} = -m\,\dot{\vec{\omega}} \times \mathbf{r}\,. \tag{3}$$

These inertial forces do not represent an indispensable concept and it has even been suggested that they should not be introduced, at least in first courses (see, for instance, [6]). However, after dealing with a few examples and practical problems (Foucault pendulum, projectile deflection, hurricanes,...) students come to appreciate that, far from getting into a muddle, the use of inertial forces can lead to a much simpler and more intuitive description of rotating systems [7].

In spite of its obvious advantages, most authors, with a few exceptions (more notoriously [8]), have avoided to extrapolate this idea into Lagrange's Mechanics through the introduction of velocity-dependent potentials. Nevertheless, such a generalization is simple enough [9] and can be used to ease the description of rotating systems that are rather tricky to analyze when studied from an inertial frame of reference. Furthermore, students find that its understanding can be greatly eased by a one-to-one analogy with a particular electromagnetic field of symmetric gauge [9].

In what follows, after a concise review of the concept of velocity-dependent and non-inertial potentials, we derive a velocity-dependent potential for a rigid body in a rotating frame. Its relation to the electromagnetic case is also addressed, identifying the mechanical analogs of known electromagnetic objects. Finally, we describe a few examples of how this non-inertial potential can be used as an alternative description of rigid body problems.

II. VELOCITY-DEPENDENT POTENTIAL

Let us consider the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{T}}{\partial q_j} = \tilde{Q}_j \qquad j = 1, 3N - k$$

for a system of N particles of mass m_i and k holonomous constraints. The position $\mathbf{r}_i = \mathbf{r}_i(t)$ of each particle can be written in terms of the $3N-k$ generalized coordinates q_j as $\mathbf{r}_i = \mathbf{r}_i(q_1, \cdots, q_{3N-k}, t)$. Here we have defined the Kinetic Energy

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2\,,$$

and the Generalized Forces

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\,. \tag{4}$$

Whenever part of these forces can be derived from a potential function V depending only on the coordinates q_j, the velocities $\dot{q}_j$ and, eventually, the time t, as

$$Q_j = \tilde{Q}_j + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial V}{\partial q_j}\,, \tag{5}$$

the Lagrange equations can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tilde{Q}_j\,,$$

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where $\mathcal{L} = T - V$ is the *Lagrangian* function. Velocity-dependent potentials were introduced in Lagrangian mechanics by the German mathematician Ernst Christian Julius Schering (1833–1897) in 1873 [10] as a way of dealing with the pre-Maxwellian Electrodynamical Theory of Wilhelm Eduard Weber (1804–1891) [11]. It was coined as Schering potential by Edmund Taylor Whittaker (1873–1956) in the first edition of his Analytical Dynamics [12], but he dropped this attribution in later editions (see [13]).

III. VELOCITY-DEPENDENT POTENTIAL OF AN ELECTROMAGNETIC FIELD

Far from being academic, the idea of a velocity-dependent potential can serve as a path for solving inverse problems [14] and comprises the case of a particle of charge q in the presence of an electromagnetic field $[\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)]$. The Lorentz Force $\mathbf{F} = q\,(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$ can be written in terms of the scalar (ϕ) and vectorial ($\mathbf{A}$) potentials (such that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial A/\partial t$) as

$$\mathbf{F} = q\left(-\nabla\phi - \frac{\partial A}{\partial t} + \dot{\mathbf{r}} \times \nabla \times \mathbf{A}\right) . \tag{6}$$

By replacing in Eq. (4) the corresponding generalized forces can be written as in Eq.(5) when the following velocity-dependent potential is defined

$$V = q\,\phi - q\,\dot{\mathbf{r}} \times \mathbf{A}\,. \tag{7}$$

IV. NON-INERTIAL POTENTIAL

Let us now consider a particle of mass m located at a position $\mathbf{r}$ in a reference frame rotating with velocity $\vec{\omega} = \vec{\omega}(t)$. Some simple algebra shows that the three fictitious forces Eqs. (1–3) can be rewritten as

$$\begin{aligned}\mathbf{F}_\mathrm{I} &= \mathbf{F}_\mathrm{centrifugal} + \mathbf{F}_\mathrm{Euler} + \mathbf{F}_\mathrm{coriolis} \\ &= \frac{m}{2}\nabla\,\left(\vec{\omega} \times \mathbf{r}\right)^2 - m\,\frac{\partial}{\partial t}\left(\vec{\omega} \times \mathbf{r}\right) \\ &\qquad + m\dot{\mathbf{r}} \times \left(\nabla \times (\vec{\omega} \times \mathbf{r})\right) .\end{aligned}$$

This expression is identical to that of the Lorentz force (6), whenever the mass is identified with the charge, $q \leftrightarrow m$, and the following scalar and vectorial potentials are defined [1, 9, 15]

$$\phi = -\,\frac{1}{2}\,\left(\vec{\omega} \times \mathbf{r}\right)^2 \qquad , \qquad \mathbf{A} = \vec{\omega} \times \mathbf{r}\,,$$

Note, in particular, that this analogy is valid whenever $\mathbf{B}$ is uniform ($\mathbf{B} = 2\vec{\omega}$) and satisfies the Symmetric Gauge $\mathbf{A} = -\mathbf{r} \times \mathbf{B}/2$ [16]. The corresponding Electric field has a very particular linear dependence on the position $\mathbf{r}$,

$$\begin{aligned}\mathbf{E} &= -4\,\mathbf{B} \times (\mathbf{B} \times \mathbf{r}) - 2\,\dot{\mathbf{B}}\,\times \mathbf{r} \\ &= 4\,B^2\,\mathbf{r} - 4\,(\mathbf{B}\cdot\mathbf{r})\,\mathbf{B} - 2\,\dot{\mathbf{B}}\,\times \mathbf{r}\,.\end{aligned}$$

Replacing in Eq. (7) we obtain the following inertial potential [9]

$$V_\mathrm{I} = -\frac{m}{2}\,\left(\vec{\omega} \times \mathbf{r}\right)^2 - m\,\dot{\mathbf{r}}\cdot(\vec{\omega} \times \mathbf{r})\ .$$

Thus, the Lagrangian for a system of N particles of masses m_i and positions $\mathbf{r}_i = \mathbf{r}_i(t)$ reads [8]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\left(\vec{\omega} \times \mathbf{r}_i\right)^2 \\ &\qquad + \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i\cdot(\vec{\omega} \times \mathbf{r}_i) - V \\ &= \frac{1}{2}\sum_{i=1}^{N} m_i\left(\dot{\mathbf{r}}_i + \vec{\omega} \times \mathbf{r}_i\right)^2 - V\ ,\end{aligned}$$

where $\vec{\omega} \times \mathbf{r}_i$ is the drift velocity of each particle in the rotating reference frame [9]. These equations can be written in terms of general coordinates $\{q_k\}_{j=1}^{3N-k}$ once the relations $\mathbf{r}_i = \mathbf{r}_i(q_1, \cdots, q_{3N-k}, t)$ are established.

V. INERTIAL POTENTIAL FOR A RIGID BODY

Let us now apply these ideas to a rigid body consisting of N point masses m_i located at $\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R}$ in a reference frame rotating with velocity $\vec{\omega} = \vec{\omega}(t)$. $\mathbf{R}$ is the position of the center of mass. The corresponding inertial potential

$$V_\mathrm{I} = -\frac{1}{2}\sum_{i=1}^{N} m_i\left(\vec{\omega} \times \mathbf{r}_i\right)^2 - \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i\cdot(\vec{\omega} \times \mathbf{r}_i)\ ,$$

can be written as

$$\begin{aligned}V_\mathrm{I} &= -\frac{1}{2}\,M\left(\vec{\omega} \times \mathbf{R}\right)^2 - M\dot{\mathbf{R}}\cdot(\vec{\omega} \times \mathbf{R}) \\ &\quad -\frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\cdot\vec{\omega} - \vec{\omega}\cdot\mathcal{I}\cdot\vec{\Omega}\,.\end{aligned} \tag{8}$$

Here $M = \sum m_i$ is the total mass, $\mathcal{I}$ is the inertia tensor with respect to the center of mass of elements

$$\mathcal{I}_{jk} = \sum_{i=1}^{N} m_i({r'_i}^2\delta_{jk} - r'_{ij}r'_{ik})\,,$$

and $\vec{\Omega}$ is the angular velocity of the rigid body in the rotating reference frame. Even though the third term in Eq. 8 does not depends explicitly on the angular velocity of the rigid body, it does depend on the orientation of its inertia tensor, except when it is a scalar (spherical top) . The corresponding Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,\vec{\Omega}\cdot\mathcal{I}\cdot\vec{\Omega} + \frac{1}{2}\,M\dot{\mathbf{R}}^2 - V_\mathrm{I} - V \\ &= \frac{1}{2}\,\left(\vec{\Omega} + \vec{\omega}\right)\cdot\mathcal{I}\cdot\left(\vec{\Omega} + \vec{\omega}\right) \\ &\qquad + \frac{1}{2}\,M\left(\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{R}\right)^2 - V\ .\end{aligned}$$

Note that for a rotating reference frame with its origin located at the center of mass of the rigid body, the inertial potential is a linear function of the angular velocity,

$$V_{\rm I} = -\frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\cdot\vec{\omega} - \vec{\omega}\cdot\mathcal{I}\cdot\vec{\Omega}\,.$$

These simple results do not only provide an advantageous alternative for the Lagrangian description of rigid bodies in rotating frames, but also lead to an interesting analogy between the corresponding inertial and electromagnetic forces. For instance, the first two terms in Eq. 8 can be identified with an electromagnetic field of symmetric gauge (with $\mathbf{B} = 2\vec{\omega}$) acting on a single particle of charge $Q = M$ located at the center of mass. The other two terms in Eq. 8 can be related to electric quadrupole and magnetic dipole moments. Actually, the analogy $q_i \leftrightarrow m_i$ can be extended to the identification of the moment of inertia with an electric quadrupole moment and of the angular momentum, $\mathbf{L} = \mathcal{I}\cdot\vec{\Omega}$, with a magnetic dipole moment, $\mu \leftrightarrow \mathbf{L}/2$, in such a way that $\vec{\omega}\cdot\mathcal{I}\cdot\vec{\Omega} = \mu\cdot\mathbf{B}$. Note that there is no electric dipolar moment since the expansion is around the center of mass.

VI. EXAMPLES

A. The Gyrocompass

As a first application of the previous results, let us analyze the motion a gyrocompass, i.e. an electrically powered spinning wheel mounted on gimbals, that finds true north while not being affected by surrounding metals. An early version of this gyroscope, was patented in 1885 by the Dutch inventor Marinus van den Bos, but it was not until almost two decades later that a working model was constructed by the German scientist Hermann Franz Joseph Hubertus Maria Anschütz-Kaempfe (1872 1931). The successful tests run by the German Navy attracted the attention of the American Elmer Ambrose Sperry (1860 1930), whose patenting of an improved version lead to a legal battle by Anschütz-Kaempfe. Albert Einstein, who was appointed as expert by the Royal District Court in Berlin, finally presented a report favorable to Anschütz-Kampe's claims of primacy [17].

Let us consider that the Gyrocompass rotates around its symmetry axis of moment of inertia I_3 in a reference frame fixed to the surface of the Earth. Since its angular velocity, $\dot{\psi}$, is much larger than that of the Earth, $|\omega|$, we can approximate in Eq. 8

$$V_{\rm I} \approx -I_3\omega\dot{\psi}\cos\theta\ ,$$

where θ is the orientation of the gyrocompass about the "true north". This inertial potential provides a restitutive force that, through friction forces, will eventually orient the compass's axis towards the north celestial pole. Appealing to the electromagnetic analogy, the situation is similar to that of a magnetic moment aligning with a constant magnetic field [18].

B. Kinetic Energy in an Inertial Frame: the Top

The inertial potential also provides a simple technique for evaluating the Lagrangian of a rotating body with much ease, even as seen from an inertial reference frame. Let us consider, for instance, the motion of a symmetrical top ($I_1 = I_2 \neq I_3$) balancing on one of its extremes [13]. When seen from a reference frame revolving with the precession velocity, $\vec{\omega} = \dot{\phi}\hat{z}$ (not known a priori), we can trace back to equation (8) and compute the inertial potential

$$V_{\rm I} = -\frac{1}{2}\,\omega^2\ \left[I_3\cos^2\theta + (I_1 + m\ell^2)\sin^2\theta\right] - I_3\omega\dot{\psi}\cos\theta.$$

Here θ, ϕ and ψ are the three Euler angles [13] and ℓ is the distance from the center of mass to the balancing point. This potential corrects the kinetic energy

$$T_{\rm I} = \frac{1}{2}I_3\dot{\psi}^2 + \frac{1}{2}(I_1 + m\ell^2)\dot{\theta}^2\,,$$

evaluated in the rotating frame, to yield the usual expression [13] in the inertial frame

$$\begin{aligned} T &= T_{\rm I} - V_{\rm I} \\ &= \frac{1}{2}I_3(\dot{\psi} + \omega\cos\theta)^2 + \frac{1}{2}(I_1 + m\ell^2)(\dot{\theta}^2 + \omega^2\sin^2\theta)\ . \end{aligned}$$

C. General Case: A First Integral for Rolling Bodies

As a third and final example, let us derive a first integral for the motion of an "arbitrary" (strictly) convex rigid body onto an horizontal plane that rotates with angular velocity $\vec{\omega} = \omega\hat{z}$. We describe its dynamics by means of the position $\mathbf{R}$ of its center of mass and the three Euler angles (ϕ, θ, ψ) as seen from a reference frame rotating with the plane. In general, and except for bodies with particularly strong symmetries such as disks or spheres [19–21], the distance $\mathbf{r}$ from the center of mass to the contact point with the surface is not fixed in magnitude or direction, but a function of Euler angles. This holonomous constraint leaves us with five degrees of freedom. Now let us assume that the body does not slide, but is free to roll on the surface and to rotate about its vertical axis $\hat{z}$. Thus, the motion of the body is constrained by

$$0 = \dot{\mathbf{R}} + \vec{\Omega}\times\mathbf{r}\ , \tag{9}$$

where $\vec{\Omega}$ is the angular velocity of the body. Being non-integrable, this constraint does not lead to any further reduction in the number of degrees of freedom. On the contrary, the relevant feature about it is its linearity in the generalized velocities. This is clear with respect to its dependence on $\dot{\mathbf{R}}$ and the same is valid with respect to $\vec{\Omega}$, whose dependence on $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ is also linear, even though the coefficients of these linear relations might in general be non linear functions of the coordinates. Thus, including the two Lagrange multipliers terms of the form $\sum \lambda_i f_i(q)\dot{q}_i$ that implement the constraint (9), the Lagrangian remains time independent and has the form:

$$\mathcal{L} = \sum A_i(q)\,\dot{q}_i + \sum B_{ij}(q)\,\dot{q}_i\dot{q}_j - D(q)\,.$$

So, according to the Noether theorem [22], we have the following first integral [23],

$$c = \sum_k \dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \;-\; \mathcal{L} = \sum B_{ij}(q)\,\dot{q}_i\dot{q}_j + D(q)\,.$$

This means that we only have to remove the linear terms in $\dot{q}$ from the Lagrangian and change the sign of the velocity independent term. So, writing the potential (8) as

$$\begin{aligned} V_{\mathrm{I}} \;=\; & \underbrace{-\frac{1}{2}\,M\,(\vec{\omega}\times\mathbf{R})^2 - \frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\cdot\vec{\omega}}_{velocity\,independent} \\ & -\underbrace{\vec{\omega}\cdot\mathcal{I}\cdot\vec{\Omega} - M\dot{\mathbf{R}}\cdot(\vec{\omega}\times\mathbf{R})}_{linear\,in\,velocities}\,. \end{aligned}$$

we finally obtain

$$c = \frac{1}{2}\,\vec{\Omega}\cdot\mathcal{I}\cdot\vec{\Omega} + \frac{1}{2}\,M\dot{\mathbf{R}}^2 - \frac{1}{2}\,M\,(\vec{\omega}\times\mathbf{R})^2 - \frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\cdot\vec{\omega}\,.$$

Clearly, this is the non inertial version of energy conservation. The first two terms are the standard kinetic energy, the third is an effective centrifugal potential and the last is an effective potential which depends on the orientation of the body. This last term becomes trivial only when the body has spherical symmetry.

VII. CONCLUSIONS

In this article we have discussed the use of inertial potentials in the Lagrangian analysis of rigid bodies in rotating frames. Besides describing the analogy of these potentials with a particular electromagnetic field of symmetric gauge, we presented three examples. They were chosen as diverse as possible so as to span a variety of different situations, in order to demonstrate the feasibility and advantages of this description. We conclude that this is a valid alternative to the standard Lagrangian description of rigid bodies from inertial frames of reference.

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[1] M. D. Semon and G. M. Schmieg, "Note on the analogy between inertial and electromagnetic forces," Am. J. Phys. **49**, 689–690 (1981).
[2] G. G. Coriolis, "Mémoire sur le principe des forces vives dans les mouvements relatifs des machines," Journal de l'École Polytechnique **13**, 268–302 (1832).
[3] G. G. Coriolis, "Mémoire sur les équations du mouvement relatif des systèmes de corps." Journal de l'École Polytechnique **15**, 142–154 (1835).
[4] P. S. Laplace, "Recherches sur plusieurs points du systeme du monde," Memoires de l'Academie Royale des Sciences de Paris **88**, 75–182 (1776).
[5] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer Verlag, 2nd. edition, 1999), p. 251.
[6] C. F. Hagenow, "Is There a Centrifugal Force?," Am. Phys. Teacher **3**, 190 (1935).
[7] G. D. Scott, "Centrifugal Forces and Newton's Laws of Motion," Am. J. Phys. **25**, 325 (1957).
[8] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon Press Ltd., Oxford, 1976) 3rd. edition, pp. 126–129.
[9] J. Sivardiere, "On the analogy between inertial and electromagnetic forces," Eur. J. Phys. **4**, 162–164 (1983).
[10] E. Schering, "Gött. Abh.", **18**, 3 (1873).
[11] W. Weber, "Annalen d. Phys", **LXXIII**, 193 (1848).
[12] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, 1904) 1st. edition.
[13] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1981), p. 21.
[14] M. L. Coffman, "Velocity-Dependent Potentials for Particles Moving in Given Orbits," Am. J. Phys. **20**, 195–199 (1952).
[15] R. Coisson, "On the Vector Potential of Coriolis Forces," Am. J. Phys. **41**, 585 (1973).
[16] G. Rousseaux, "The gauge non-invariance of Classical Electromagnetism," Annales de la Fondation Louis de Broglie **30**, 387–396 (2005).
[17] M. B. May, "Elmer A. Sperry and the Gyrocompass," The Institute of Navigation Newsletter **17** (2), 8–9 (2007).
[18] G. I. Opat, "Coriolis and magnetic forces: The gyrocompass and magnetic compass as analogs," Am. J. Phys. **58**, 1173–1176 (1990).
[19] J. Barcelos-Neto and M. B. Dias da Silva, "An example of motion in a rotating frame," Eur. J. Phys. **10**, 305–308 (1989).
[20] K. Weltner, "Stable circular orbits of freely moving balls on rotating discs," Am. J. Phys. **47**, 984–986 (1979).
[21] A. Bloch, P. Krishnaprasad, J. Marsden and R. Murray, "Nonholonomic Mechanical Systems with Symetry", Technical Report. California Institute of Technology.

http://resolver.caltech.edu/CaltechCDSTR:1995.CIT-CDS-94-013 (1995).

[22] E. Noether, “Invariante Variationsprobleme”, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, 235-257 (1918).

[23] E. A. Desloge and R. I. Karch, “Noether’s theorem in classical mechanics,” Am. J. Phys. **45**, 336–339 (1976).