Entanglement and irreversibility on the light-cone

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QFT has emerged as the framework for quantum many body systems, in high energy and CM physics.

Fundamental problem: given a microscopic QFT, find its long distance dynamics

⇒ **Systematic approach: the renormalization group (RG)**

- Physics at a scale $\Lambda$ described by EFT.
- Integrate out d.o.f. with $E > \Lambda$
- Produces a flow in the space of couplings

$$E \frac{dg_I}{dE} = \beta_I(g)$$
Think about “space of QFTs”, RG flows & fixed points

Crucial property: *irreversibility of the RG*

Intuition: Loss of information about UV d.o.f.

**E.g. C-thm**

“Irreversibility” of the flux of the renormalization group in a 2D field theory

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Results on RG evolution as dissipative process:

0 + 1 ✓ *g*-theorem [Affleck, Ludwig; Friedan, Konechny …]
1 + 1 ✓ *c*-theorem [Zamolodchikov; Cappelli, Friedan …]
2 + 1 ✓ *F*-theorem [Myers, Sinha; Jafferis et al; Casini, Huerta]
3 + 1 ✓ *a*-theorem [Cardy; Komargodski, Schwimmer …]
any D ✓ holographic C-thms [Freedman et al; Myers, Sinha, …]

Use quite different tools: unitarity, dilaton, entanglement …

⇒ Underlying principle for irreversibility of the RG?
⇒ Useful way of comparing QFTs, and defining distance in theory space?
⇒ What happens in D>4?
Goal of the talk:

Review recent progress on understanding the RG using quantum information theory.

Based on collaborations with Horacio Casini, Ignacio Salazar & Eduardo Testé, at Bariloche

Outline:

A. Relative entropy on the light-cone
B. Irreversibility & Markov property
C. Work in progress & future directions
A. Relative entropy on the light-cone

- Need: measure for \( N_{\text{dof}} \) along the RG & way of comparing theories at different scales

- QIT offers an appropriate framework for this. To make progress, start from the simplest possible RG:

**Boundary RG flows**

2d CFT

BCFT characterized by \( g \);
measures \( N_{\text{dof}} \) at boundary

\[
S_{\text{thermal}} = \frac{\pi c}{3} TL + \log g
\]

E.g. Kondo model

[Casini, Salazar, GT 2016]
“g-theorem”: $g$ decreases along boundary RG flows

RG due to relevant boundary deformation

$$S = S_{BCFT} + \int dx_0 \lambda \mathcal{O}(x)$$

$$\log g_{UV} > \log g_{IR}$$

[Affleck, Ludwig; Friedan, Konechny]

Understand using QIT? $g$ measured by EE:  

[Cardy]

$$S(r) = \frac{c}{6} \log \frac{r}{\epsilon} + \log g$$

$$\Rightarrow \log \frac{g_{IR}}{g_{UV}} = S_{IR}(r) - S_{UV}(r)$$

Monotonicity properties of $\Delta S$?
Suggests using the relative entropy!

Define the reduced density matrices on interval $r$:

$$\sigma = \text{tr}_V |0\rangle\langle 0| : \text{UV BCFT}$$

$$\rho : \text{theory w/relevant deformation}$$

Same operator content. Evolve w/different action

Then

$$S_{\text{rel}}(\rho|\sigma) = \text{tr}(\rho \log \frac{\rho}{\sigma})$$

✓ measure of distance between states

✓ Central in QIT and various physics appl's

CFT modular Hamiltonian

$$H = -\log \sigma$$

$$\text{tr}(\rho H) - \text{tr}(\sigma H)$$

$$S(\rho) - S(\sigma) = \log \frac{g(r)}{g(0)}$$

Irreversibility of the RG, measured by $\Delta S$, could then follow from monotonicity and positivity of relative entropy. But ...

This simple idea does not quite work: generically $\Delta \langle H \rangle \gg \Delta S$
Then try to minimize $\Delta \langle H \rangle$. For this, change Cauchy surface:

$$\Delta S \text{ is indep of } \Sigma$$

but

$$\Delta H = \int_{\Sigma} ds \, \eta^\mu \xi^\nu \Delta \langle T_{\mu\nu} \rangle$$

normal to $\Sigma$

CKV that keeps interval fixed

• **static limit** $x^0 = 0$, $H = 2\pi \int_0^r dx^1 \frac{r^2 - (x^1)^2}{2r} T_{00}(x^1)$

Since $\Delta \langle T_{00} \rangle \sim \delta(x^1) \Rightarrow \Delta \langle H \rangle \sim r$ relative entropy distinguishes too much

• **null limit** $x^- = -r$, $H = 2\pi \int_{-r}^r dx^+ \frac{r^2 - (x^+)^2}{2r} T_{++}(x^+)$

$\Delta \langle T_{++} \rangle \sim \delta(x^+ + r) \Rightarrow \Delta \langle H \rangle = 0$ boundary sits in high temperature region

Then $S_{\text{rel}}(\rho_r | \sigma_r) = -\log \frac{g(r)}{g(0)}$ & $\frac{dS_{\text{rel}}(r)}{dr} \geq 0 \Rightarrow g'(r) \leq 0$
Apply this approach to RG flows in d-dims

\[ S = S_{\text{CFT}} + \int d^d x \lambda \mathcal{O}(x) \]

\[ \text{dim } \Delta < d \]

Compare both theories using \( S_{\text{rel}} \) and minimize \( \Delta \langle H \rangle \) by taking the null limit

properties of CFT stress tensor imply

\[ \Delta \langle T_{\mu \nu} \rangle \Sigma \sim \lambda^2 \epsilon^{d-2\Delta} (\eta_{\mu} \eta_{\nu} - \frac{g_{\mu \nu}}{d}) \]

\[ \Rightarrow \Delta \langle H \rangle \Sigma \sim \lambda^2 \epsilon^{d-2\Delta} \int_{\Sigma} \eta^\mu \xi_\mu \]

- \( \Delta \langle H \rangle_{x^0 \text{ const}} \sim \lambda^2 \epsilon^{d-2\Delta} r^d \)
- \( \Delta \langle H \rangle_{\text{null}} \sim \lambda^2 \epsilon^{d+2-2\Delta} r^{d-2} \) area law! (high T)
For $\Delta < \frac{d+2}{2}$, $\Delta \langle H \rangle_{\text{null}} = 0 \Rightarrow S_{\text{rel}}(\rho_r | \sigma_r) = S(\sigma_r) - S(\rho_r)$

Monotonicity of relative entropy then implies:

• $d = 2$ : $S_{\text{rel}}(r) \approx \frac{1}{3}(c_{UV} - c_{IR}) \log(mr) \Rightarrow c_{UV} > c_{IR}$

C-theorem

• $d > 2$ : $S_{\text{rel}}(r) \approx (\mu_{UV} - \mu_{IR}) r^{d-2} \Rightarrow \mu_{UV} > \mu_{IR}$

$$\approx -\Delta \left( \frac{1}{4G_N} \right) r^{d-2}$$

“area theorem”

Lessons so far:

✓ $S_{\text{rel}}$ on light-cone provides useful statistical distance in QFT
✓ Quantum-information meaning for $\Delta g$, $\Delta c$, $\Delta \mu$
✓ Irreversibility of RG in terms of increased distinguishability or information loss
B. Irreversibility & Markov property

[Casini, Testé, GT, 2017]

For a CFT in $d$-dims, consider the EE on a sphere:

$$S(r) = \mu_{d-2} r^{d-2} + \mu_{d-4} r^{d-4} + \ldots + \left\{ (-1)^{d/2-1} 4A \log\left(\frac{r}{\epsilon}\right) \right\} (-1)^{[(d-1)/2]} F$$

Irreversibility of RG: look for inequalities in $\Delta A$, $\Delta F$

Requires positivity of higher derivatives of $\Delta S = S(\rho) - S(\sigma)$

Suggests looking at multiple regions. We now argue that for a CFT and regions with surface on light-cone, the SSA is saturated

$$S_\sigma(A) + S_\sigma(B) - S_\sigma(A \cap B) - S_\sigma(A \cup B) = 0$$

$$\Rightarrow \Delta S(A) + \Delta S(B) - \Delta S(A \cap B) - \Delta S(A \cup B) \geq 0$$
Focus first on null plane and then map to light-cone

Quick argument: entropy $S_\gamma$ for region w/boundary $x^+ = \gamma(y)$ should be invariant under boosts $x^+ \rightarrow \lambda x^+$.

Taking $\lambda \rightarrow 0 \Rightarrow S_\gamma$ indep of $\gamma \therefore S_{\gamma_A} + S_{\gamma_B} = S_{\gamma_A \cup \gamma_B} + S_{\gamma_A \cap \gamma_B}$
Conformal map to light-cone:

$$r^- = r - t = \gamma(\Omega)$$

- Only dependence on the curve can come from the cutoff
- This has to be local and extensive
- Terms classified using Lorentz inv

$$\Rightarrow S_\gamma = \int d^{d-2}\Omega \ f(\gamma(\Omega), \epsilon)$$

e.g. $d=4 = \int d^2\Omega \left\{ \alpha_1 \frac{\gamma^2}{\epsilon^2} + \alpha_2 \left( \log \frac{\gamma}{\epsilon} - \frac{1}{2} \left( \frac{\nabla \gamma}{\gamma} \right)^2 \right) + \ldots \right\}$

Hence ray by ray the terms in the SSA inequality cancel out and we get a saturation of SSA.
Markov property of the vacuum

We now discuss a complementary perspective

**SSA saturation** ⇔ \[ \log \rho_{A \cup B} = \log \rho_A + \log \rho_B - \log \rho_{A \cap B} \]

✓ This is called a quantum Markov state
✓ Tracing out a subsystem becomes a reversible process
✓ Roughly, no entanglement over different null lines

• Markov property also follows from result for modular Hamiltonian

\[ H_\gamma = 2\pi \int d^{d-2}y \int dx^+ (x^+ - \gamma(y))T_{++} \]

Rindler result, null line by line
Follows from OPE of twist operators and from algebraic QFT methods

See also [Lashkari; Faulker et al]
Using the geometric setup of [Casini, Huerta, 2012], consider SSA for $\Delta S'$ for boosted spheres on light-cone

- $N \to \infty$ boosted spheres
- intersections and unions give “wiggly” spheres.
- Problem: unlike $d=3$, for general $d$
  \[
  \lim_{N \to \infty} S'_{\text{wiggly}} \neq S'_{\text{sphere}} \quad \text{in SSA combination}
  \]

Crucial role of Markov property: in SSA formula

Differences between wiggly and regular spheres are UV as $N \to \infty$
Hence in SSA for $\Delta S'$, $\Delta S'_{\text{wiggly}} \to \Delta S'_{\text{sphere}}$
Repeated application of SSA:

\[
\Delta S(\sqrt{r R}) \geq \frac{1}{N} \sum_{k=1}^{N} \Delta S_k \approx \int_{r}^{R} d\ell \beta(\ell) \Delta S(\ell)
\]

union of intersection of k spheres
density of spheres of radius \( \ell \)

As \( R \to r \), \( r \Delta S''(r) - (d - 3) \Delta S'(r) \leq 0 \)

Unifies \( c, F \) and \( a \) theorems, and predicts new inequalities in higher \( d \)

Recall

\[
S(r) = \mu_{d-2} r^{d-2} + \mu_{d-4} r^{d-4} + \ldots + \left\{ \begin{array}{l}
(-1)^{d/2-1} 4A \log(r/\epsilon) \\
(-1)^{(d-1)/2} F
\end{array} \right.
\]

\( d = 2 : (r \Delta S'(r))' \leq 0 \Rightarrow \Delta c(r) = c(r) - c_{UV} = r \Delta S'(r) \)

\( d = 3 : \Delta S''(r) \leq 0 \Rightarrow \Delta F(r) = r \Delta S'(r) - \Delta S(r) \)

\( d = 4 : r S''(\rho) - S'(\rho) \leq \frac{8A_{UV}}{r} \Rightarrow A_{IR} \leq A_{UV} \)

entropic a-thm

\( \text{general } d : \frac{d}{dr} \left( \frac{\Delta S'}{r^{d-3}} \right) \leq 0, \lim_{r \to \infty} \Delta \mu_{d-4} \geq 0 \)

renormalization of gravitational terms
C. Future directions

• **Work in progress:** It seems previous arguments for independence on curve $\gamma(y)$ also work for all Renyi entropies:

\[
S_n(A) + S_n(B) - S_n(A \cap B) - S_n(A \cup B) = 0
\]

Much stronger than Markov. CFT vacuum behaves as product state over null cone. It is possible to characterize *all entropies* on light-cone. Consequences?

• **C-theorems in higher dimensions? Monotonic fc. in d=4?**

• **Applications of simple entanglement structure on light-cone. QNEC, generalized second law, monotones, …**