Rough interfaces
and
Elastic lines in disordered systems

1-Basic concepts
- Ferromagnetic domain wall
- Simplified models, interactions
- Random walk
- Fluctuations: Roughness

2-Theoretical formulation
- Correlation functions
- Family-Vicsek scaling
- Edwards-Wilkinson equation

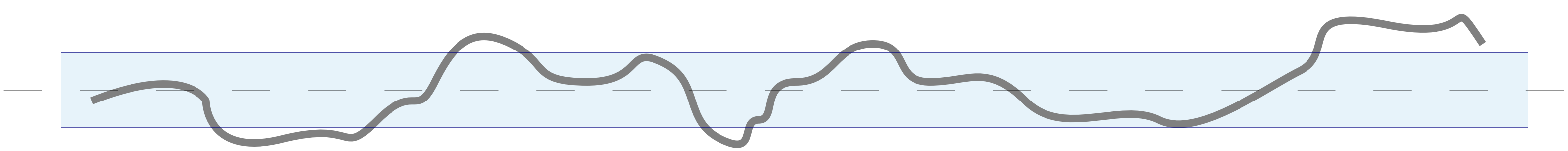
3-Universality
- Kardar-Parisi-Zhang equation
- Mullins-Herring equation
- Universality classes
- Distribution functions
- Anomalous scaling

4-Disordered media
- Quenched disorder
- Directed polymer in random media
- Ground state and transfer matrix
- Energy fluctuations
- Thermal effects

5-Depinning transition
- Experimental results
- Critical phenomena analogy
- Correlation lengths
- Dynamic regimes, thermal effects

Correlation functions

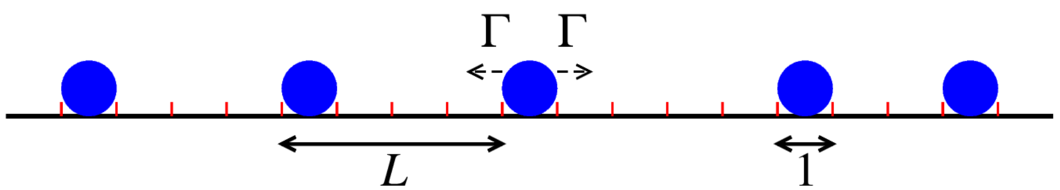
global roughness $$W^2 = \sum_{z=0}^{L-1} \left[u(z) - \langle u(z) \rangle\right]^2$$



random deposition $$W^2 \sim t$$

correlations ?????

Family Vicsek scaling



single file diffusion

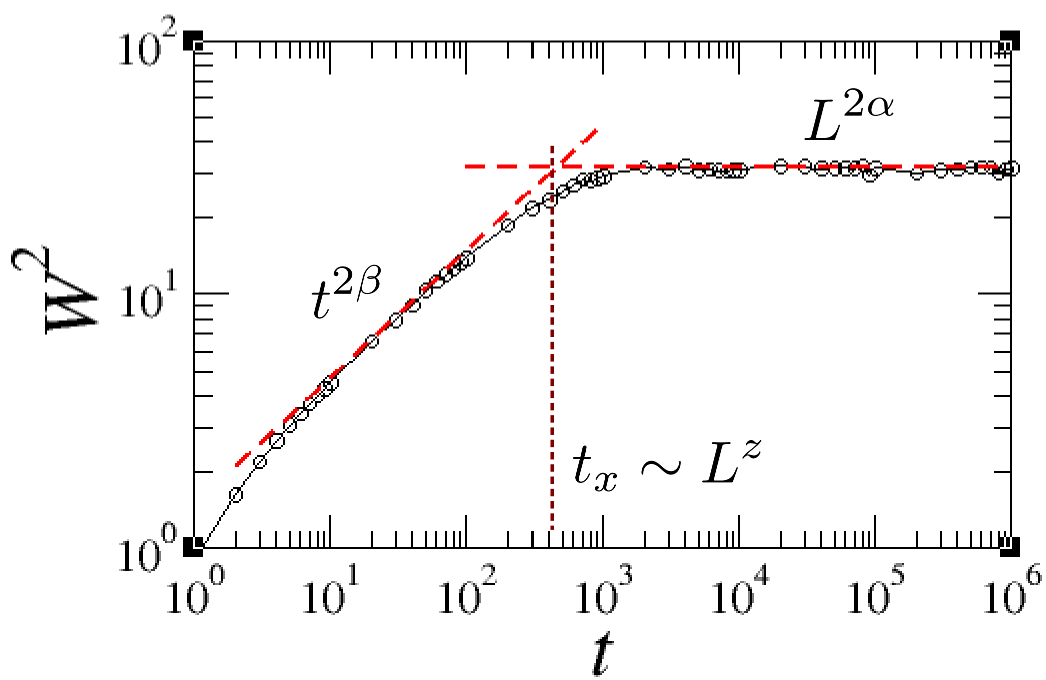
In the general case

$$W(t,L) \sim t^{\beta} \qquad t \ll t_x$$

with β the **growing exponent**

$$W(t,L) \sim L^{\alpha} \qquad t \gg t_x$$

with α the **roughness exponent**



the crossover time $t_x \sim L^z$

with z the **dynamical exponent**

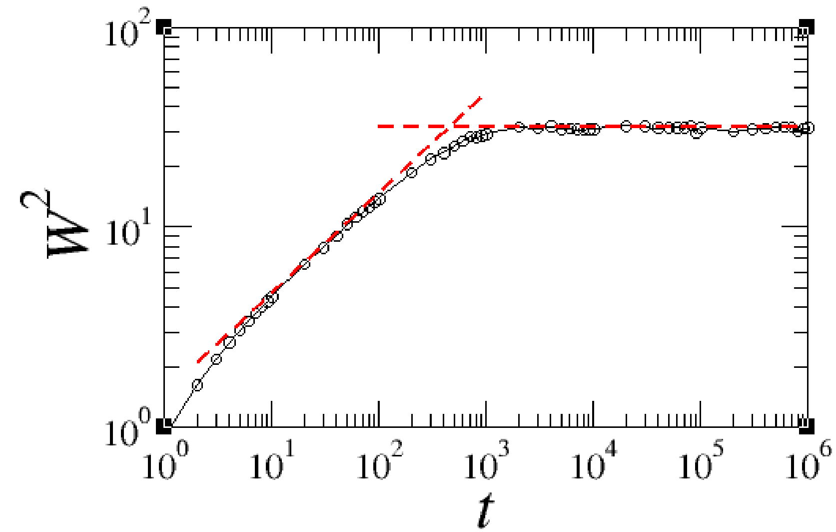
$$t_x^{\beta} \sim L^{z\beta} \sim L^{\alpha}$$

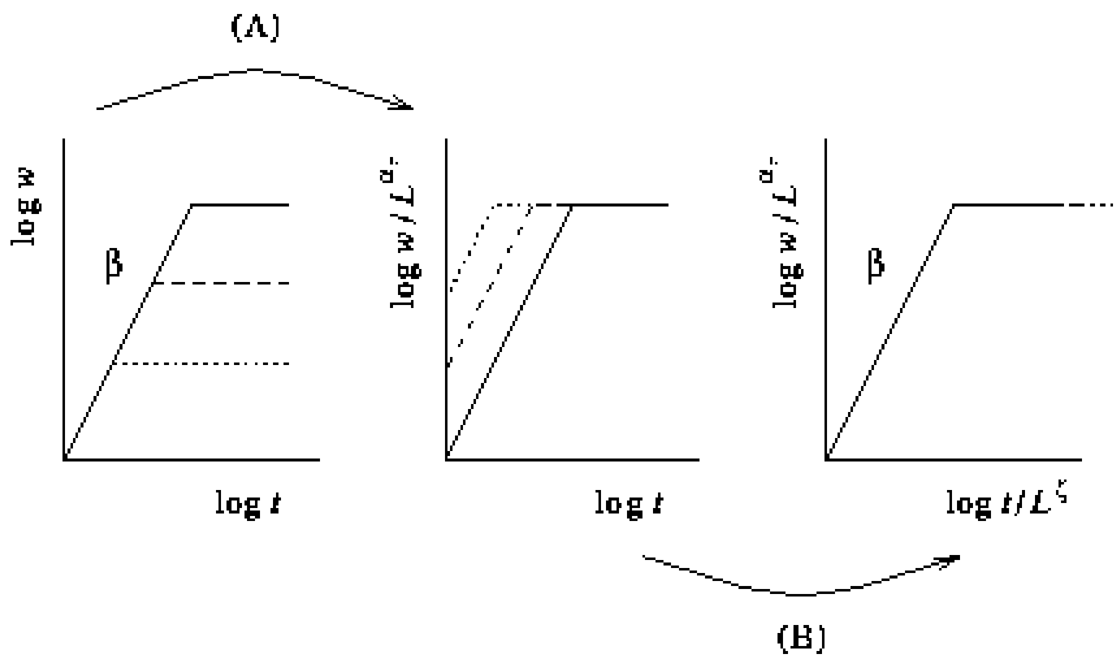
we have the scaling relation $z = \dfrac{\alpha}{\beta}$

Family Vicsek scaling

(A) Plot $\log\left(\frac{W}{L^\alpha}\right)$ vs. t

(B) Plot $\log\left(\frac{W}{L^\alpha}\right)$ vs. $\frac{t}{L^z}$





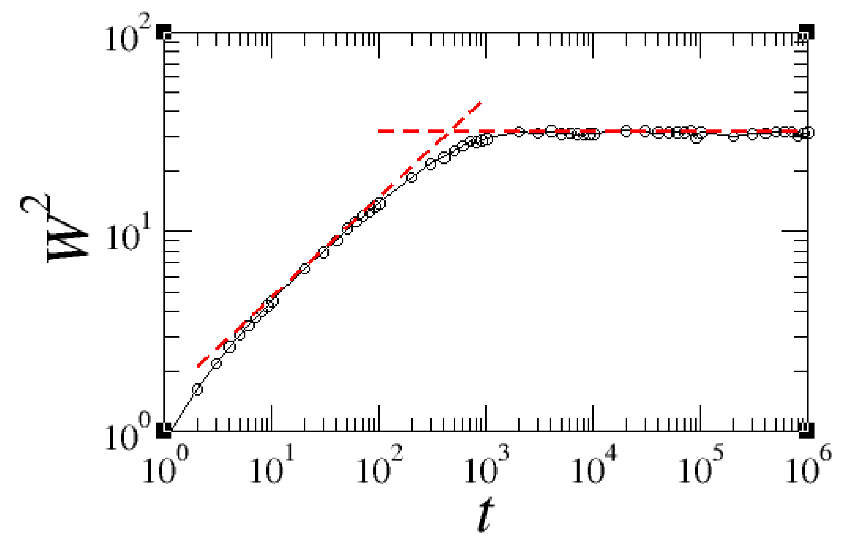
universal function

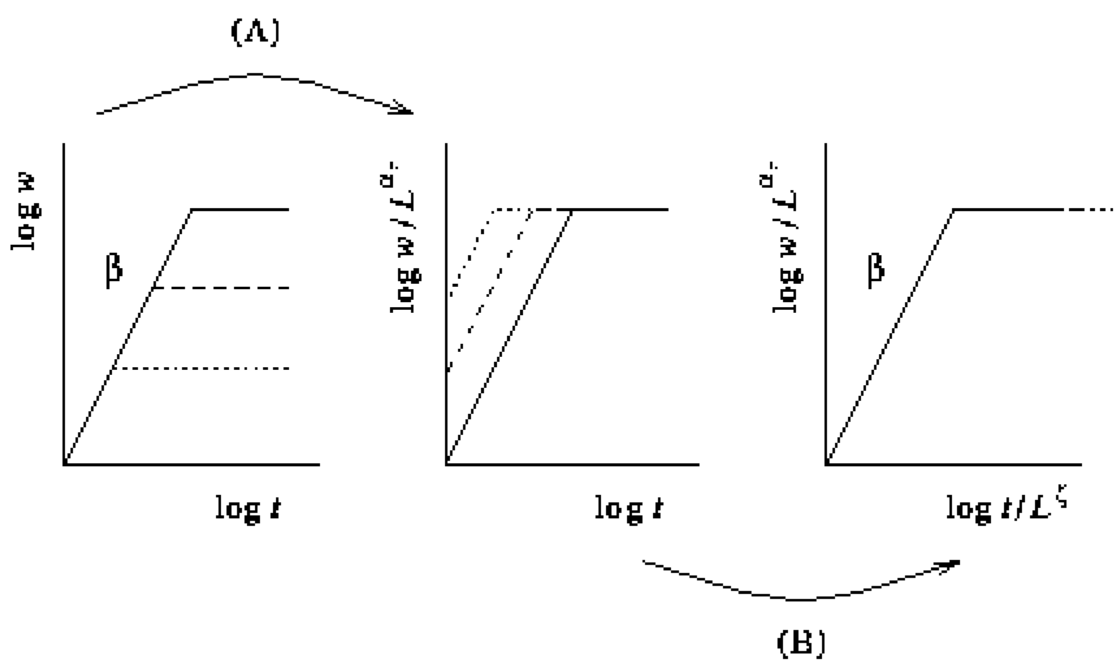
Family Vicsek scaling

$$W(L,t) \sim L^{\alpha} f\left(\frac{t}{L^{z}}\right)$$

$$f(x) \sim \begin{cases} x^{\beta} & \text{for} \quad x \ll 1 \\ 1 & \text{for} \quad x \gg 1 \end{cases}$$

scaling function





Family Vicsek scaling

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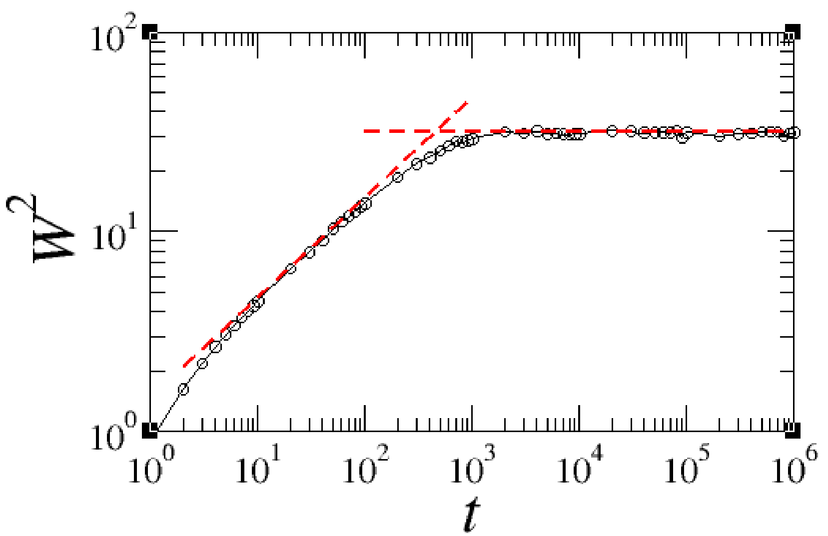
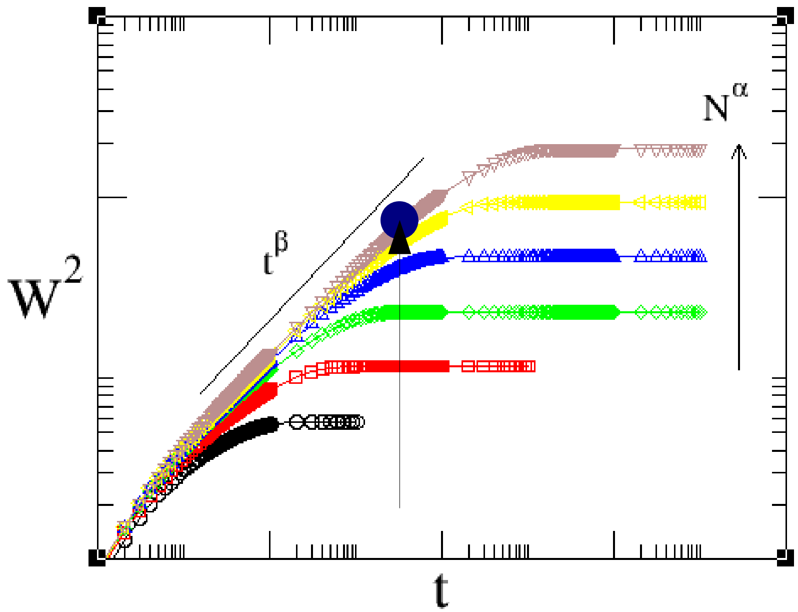
scaling function



$$W(L,t) = t^{\beta} g\left(\frac{L}{t^{1/z}}\right)$$

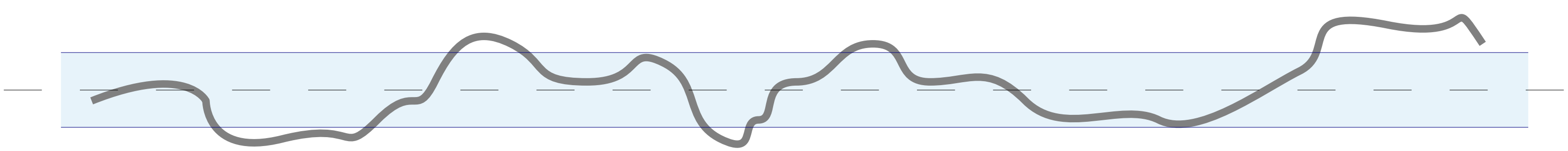
$$g(y) \sim \begin{cases} y^{\alpha} & \text{for} \quad y \ll 1 \\ 1 & \text{for} \quad y \gg 1 \end{cases}$$

transverse correlation length $\xi(t) \sim t^{1/z}$



Correlation functions

global roughness $$W^2 = \sum_{z=0}^{L-1} \left[u(z) - \langle u(z) \rangle \right]^2$$ $$W(t,L) \sim \begin{cases} t^\beta & \text{for } \xi(t) \ll L \\ L^\alpha & \text{for } \xi(t) \gg L \end{cases}$$



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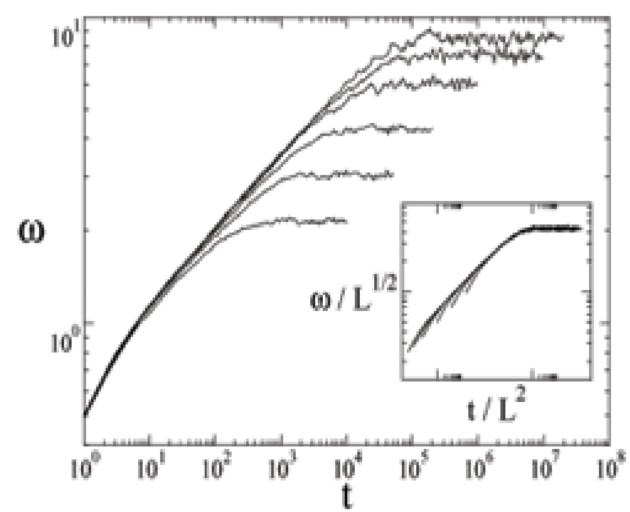
automata model within de
EW universality class



Mattos, Moreira, Atman, Brazilian J. Phys. **36**, 2006

Correlation functions

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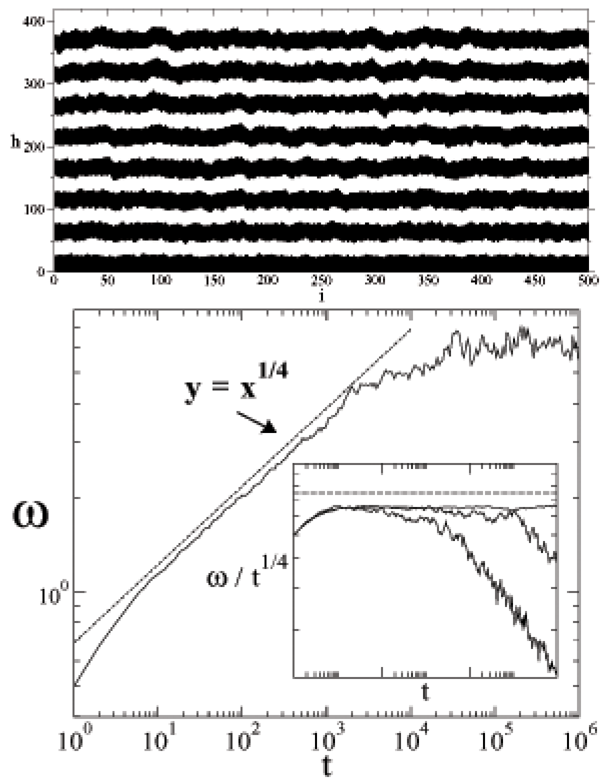
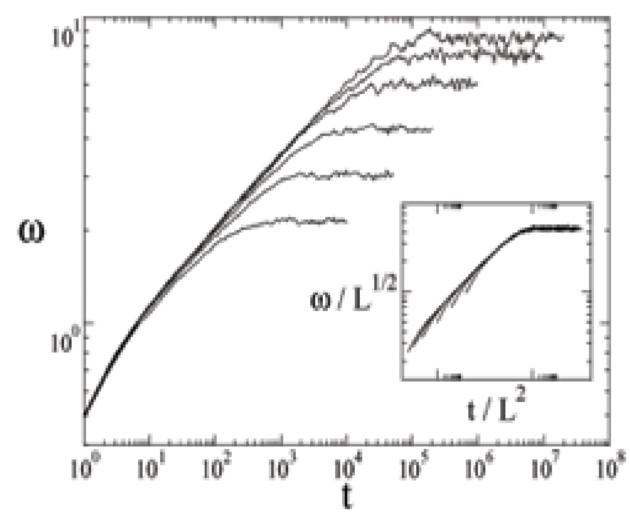


$$p_i(t) = \rho \, e^{\kappa \Gamma_i(t)}$$

$$\Gamma_i(t) = h_{i+1}(t) + h_{i-1}(t) - 2h_i(t)$$

automata model within de EW universality class



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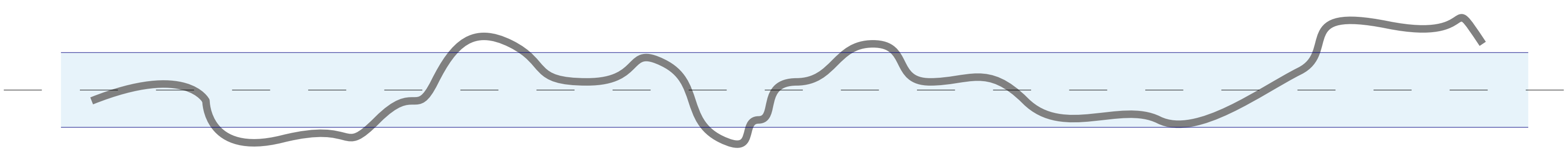
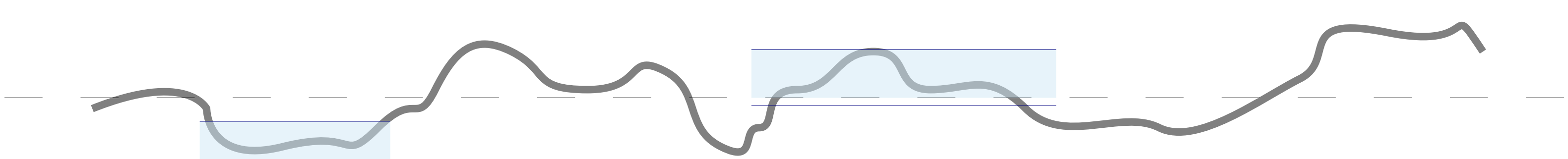


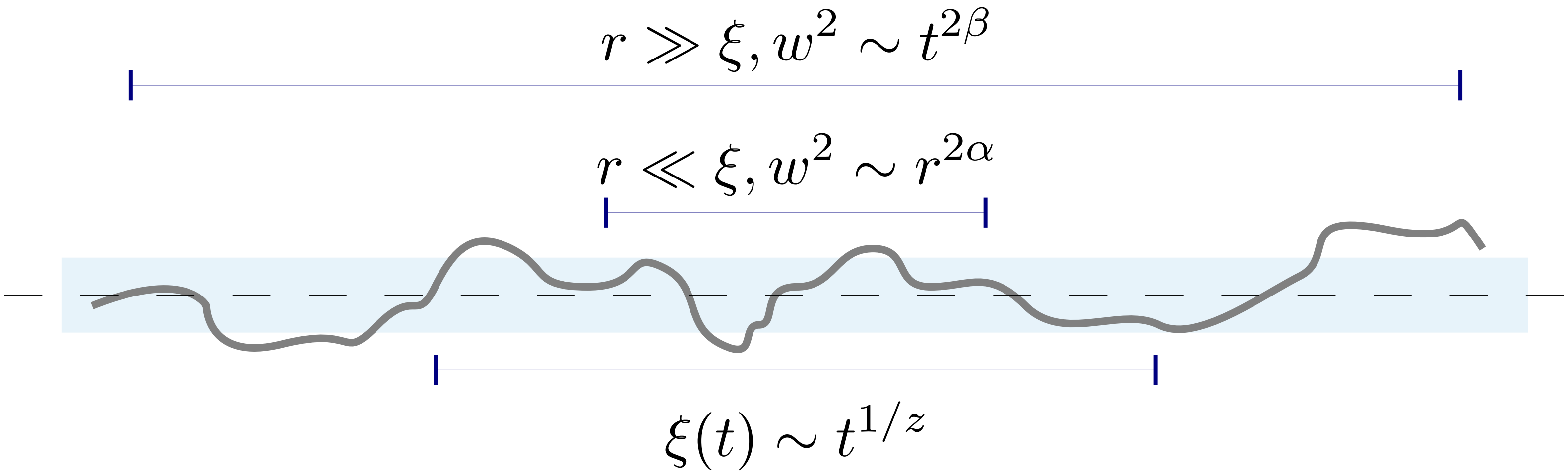
local roughness

$$w^2(r,t) = \frac{1}{L} \sum_{r'} \frac{1}{r} \sum_{z=r'}^{r'+r} \left[u(z,t) - \langle u(z,t) \rangle_r \right]^2$$

mean value over a window of size *r*



Correlation functions



local roughness

$$w^2(r,t) = \frac{1}{L}\sum_{r'}\frac{1}{r}\sum_{z=r'}^{r'+r}\left[u(z,t) - \langle u(z,t)\rangle_r\right]^2$$

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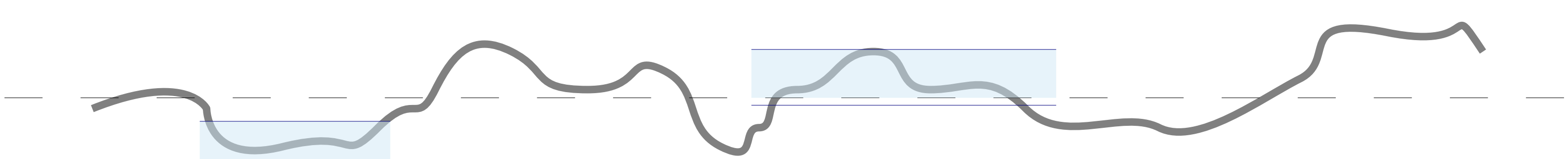
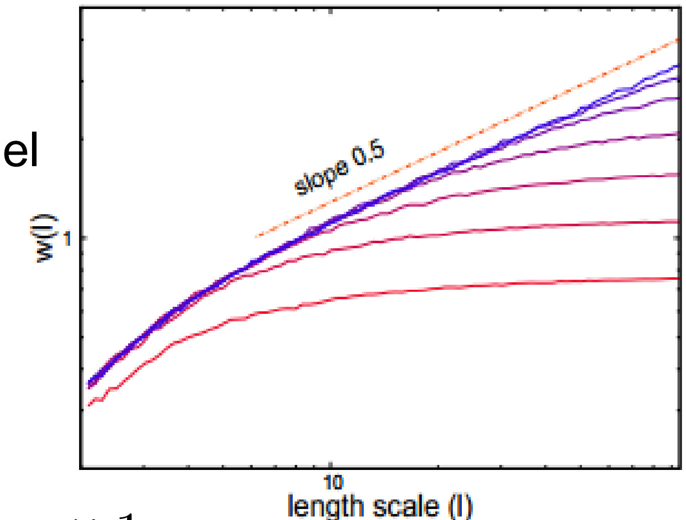


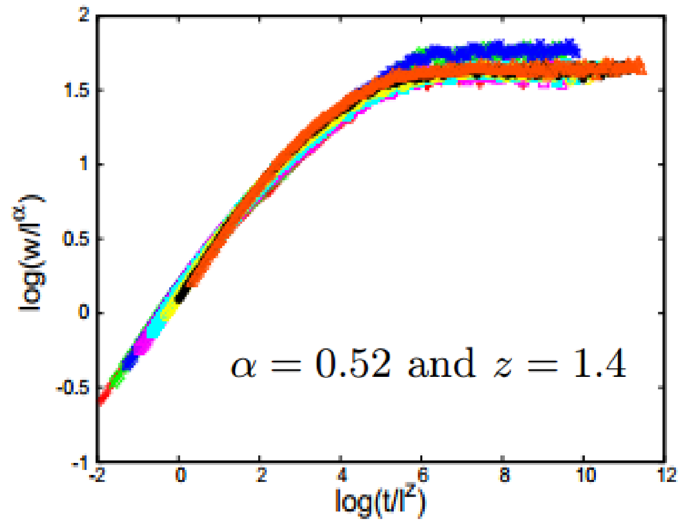
Correlation functions

intrinsic geometrical model
with normal growth

$$w(r,t) \sim r^{\alpha} f\left(\frac{t}{r^{z}}\right)$$

$$f(x) \sim \begin{cases} x^{\beta} & \text{for} \quad x \ll 1 \\ 1 & \text{for} \quad x \gg 1 \end{cases}$$



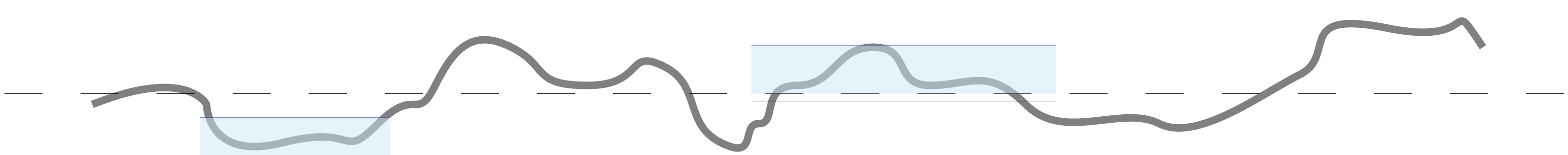


Rodríguez-Laguna, Santalla, Cuerno, JSTST, **P05032**, 2011

local roughness

$$w(r,t)^2 = \sum_{r'} \frac{1}{r} \sum_{z=r'}^{r'+r} \left[u(z,t) - \langle u(z,t)\rangle_r\right]^2$$

$$w(r,t) \sim \begin{cases} t^{\beta} & \text{for } \xi(t) \ll r \\ r^{\alpha} & \text{for } \xi(t) \gg r \end{cases}$$



Correlation functions

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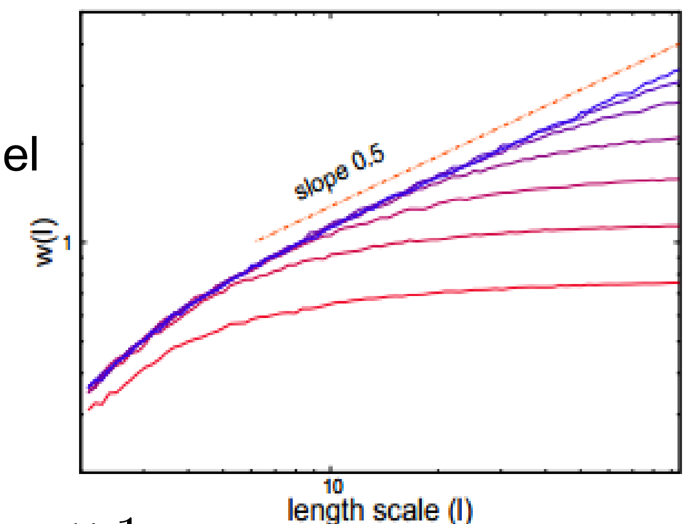
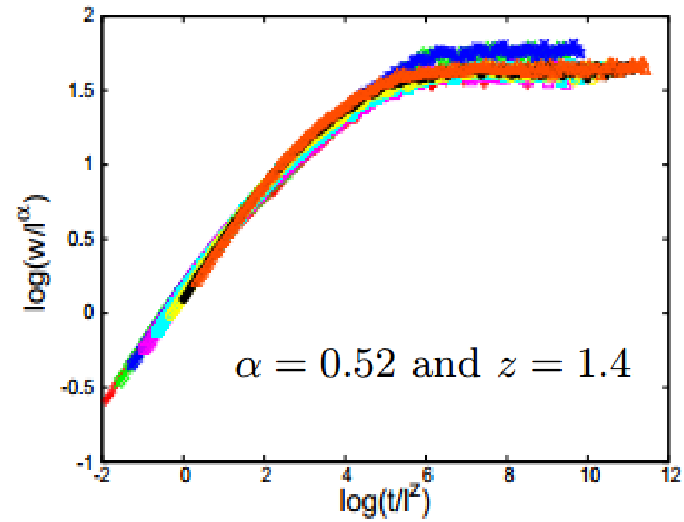


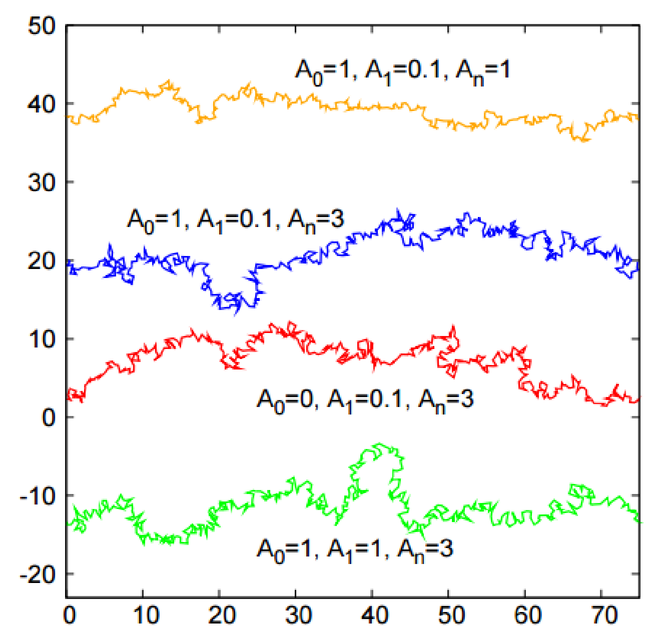


Rodríguez-Laguna, Santalla, Cuerno, JSTST, **P05032**, 2011

$$\partial_t \mathbf{r} = \left(A_0 + A_1 K(\mathbf{r}) + A_n \eta(\mathbf{r},t)\right) \mathbf{u}_n$$

local curvature

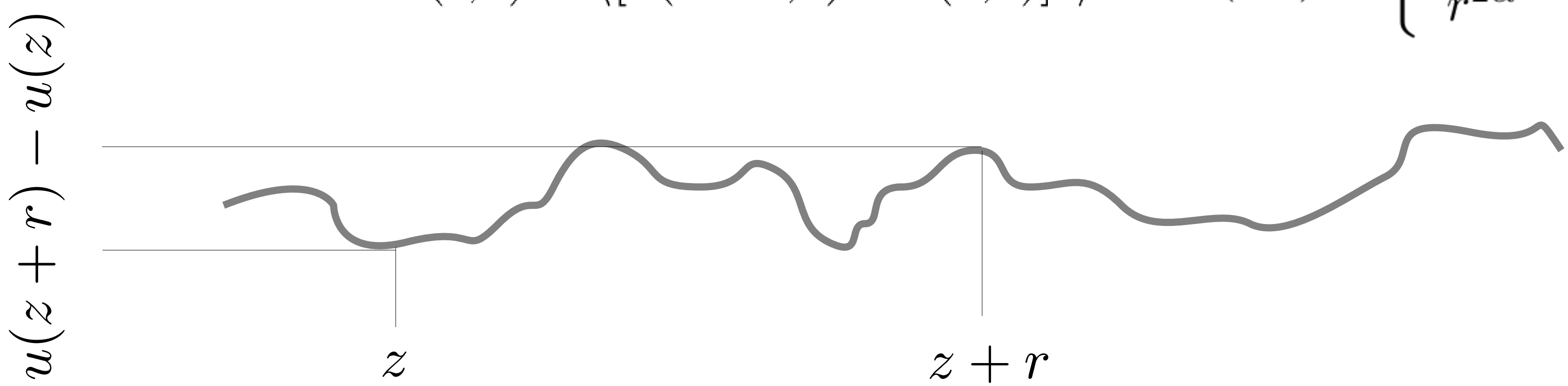
normal direction

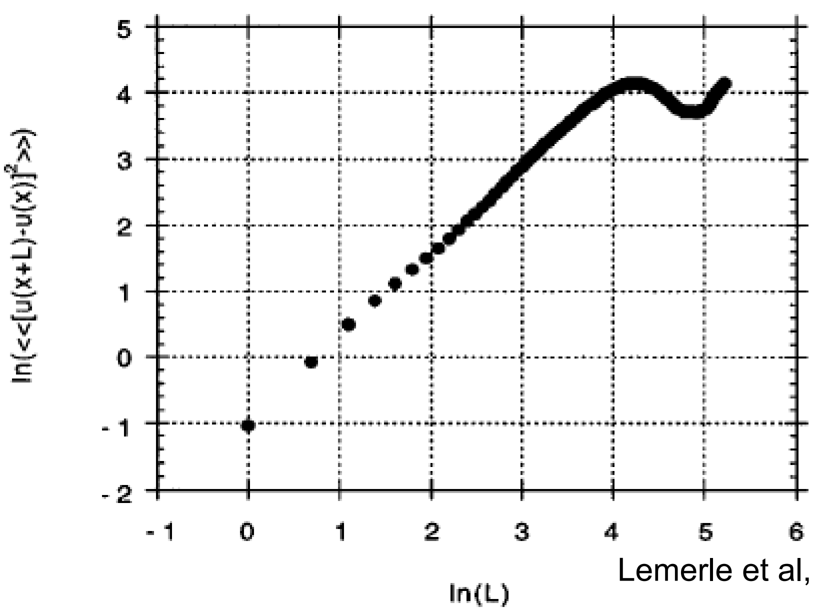


Correlation functions

displacement-displacement correlation function
(height-height correlation function)

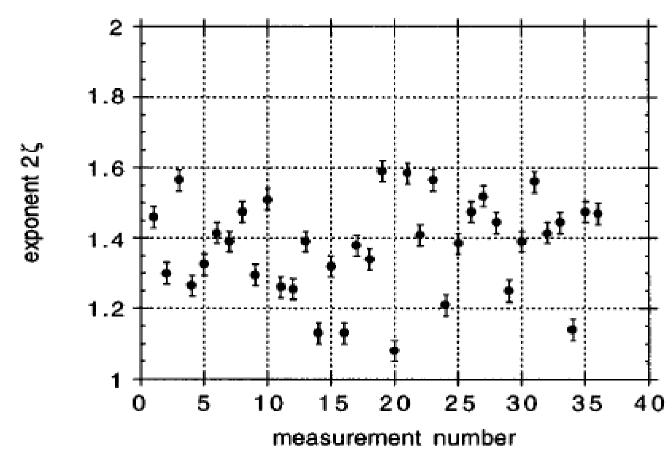
$$B(r,t) = \langle [u(z+r,t) - u(z,t)]^2 \rangle \qquad B(r,t) \sim \begin{cases} t^{2\beta} & \text{for } \xi(t) \ll r \\ r^{2\alpha} & \text{for } \xi(t) \gg r \end{cases}$$





Lemerle et al, PRL, **80**, 849 1998

experiments on ferromagnetic domain wall motion



Correlation functions

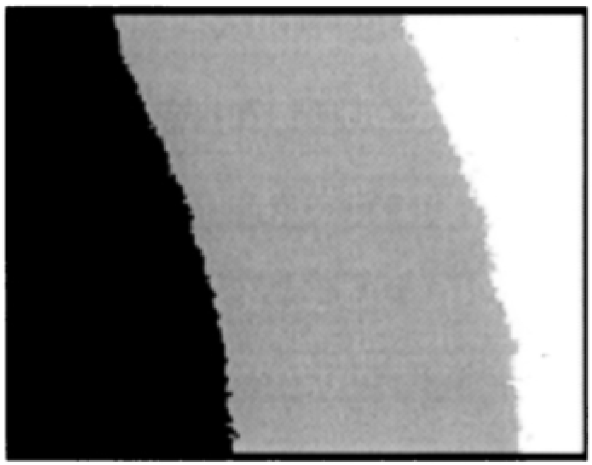


FIG. 1. Typical magneto-optical image (size $90 \times 72\ \mu\text{m}^2$, $\lambda = 638.1$ nm). The gray part corresponds to the surface swept by the domain wall during 111 μs at 460 Oe ($T = 23\,^\circ$C). The dark part is the original domain.

displacement-displacement correlation function
(height-height correlation function)

$$B(r,t) = \langle [u(z+r,t) - u(z,t)]^2 \rangle$$

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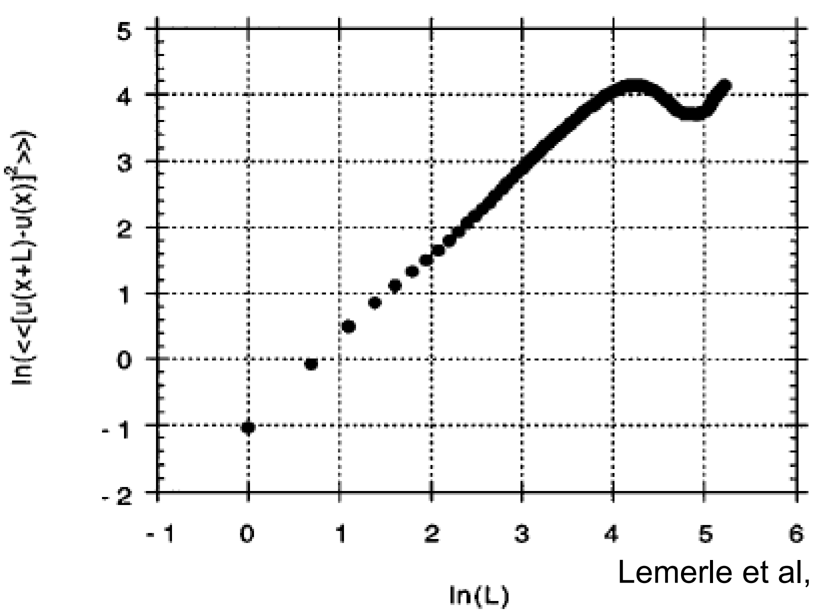

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experiments on ferromagnetic
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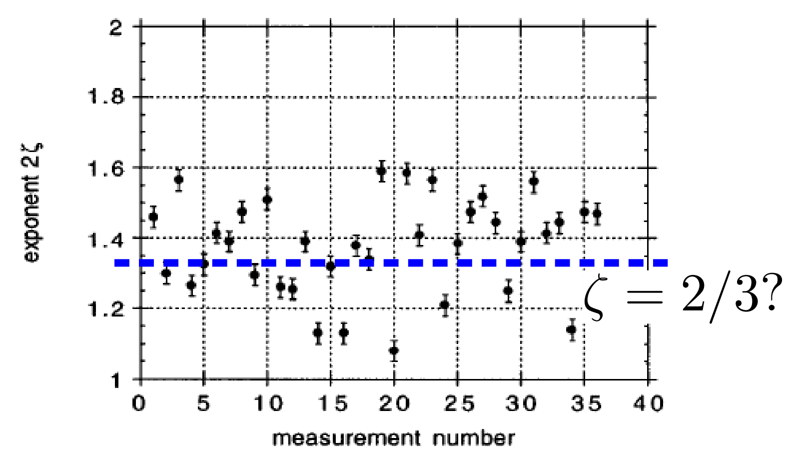

Correlation functions

structure factor

$$S(q,t) = \langle u(q,t)u(-q,t)\rangle \qquad \text{with} \qquad u(q,t) = \int dz\, u(z,t)\, e^{-iqz}$$

small q → large r

large q → small r

$$B(r,t) \propto \int_{2\pi/L}^{2\pi/a} dq[1-\cos(qr)]S(q,t)$$

a small scale cut-off

L large scale cut-off

Correlation functions

structure factor

$$S(q,t) = \langle u(q,t)u(-q,t)\rangle \quad \text{with} \quad u(q,t) = \int dz\, u(z,t)\, e^{-iqz}$$

$$B(r,t) \sim \int dq(1-\cos qr)S(q,t)$$

$$\sim \int \frac{dx}{r}(1-\cos x)S(x/r,t) \qquad x = qr$$

$$\sim \int \frac{dx}{r}(1-\cos x)\frac{x^y}{r^y} \qquad r \ll \ell \to S(q,t) \sim q^y \quad (q \gg 1/\ell)$$

$$\sim r^{-(1+y)} \int dx(1-\cos x)x^y \sim r^{-(1+y)} I$$

$$B(r) \sim r^{2\alpha} \sim r^{-(1+y)} \qquad y = -(1+2\alpha) \qquad q \gg 1/\ell \to S(q,t) \sim q^{-(1+2\alpha)}$$

Correlation functions

structure factor

$$S(q,t) = \langle u(q,t)u(-q,t)\rangle \qquad \text{with} \qquad u(q,t) = \int dz\, u(z,t)\, e^{-iqz}$$

$$q \gg 1/\ell \to S(q,t) \sim q^{-(1+2\alpha)}$$

$$S(q,t) \sim q^{-(1+2\alpha)}F(q\ell) \qquad q\ell \gg 1 \to S(q,t) \sim q^{-(1+2\alpha)}$$

$$\sim q^{-(1+2\alpha)}F(qt^{1/z}) \qquad r \gg \ell \quad (q \ll 1/\ell \sim t^{-1/z})$$

$$\sim q^{-(1+2\alpha)}(qt^{1/z})^{1+2\alpha}$$

fluctuations independent of $r(q)$

$$\sim t^{(1+2\alpha)/z}$$

$$S(q,t) \sim \begin{cases} t^{(1+2\alpha)/z} & \text{for} \quad q \ll 1/\ell \sim t^{-1/z} \\ q^{-(1+2\alpha)} & \text{for} \quad q \gg 1/\ell \sim t^{-1/z} \end{cases}$$

Correlation functions

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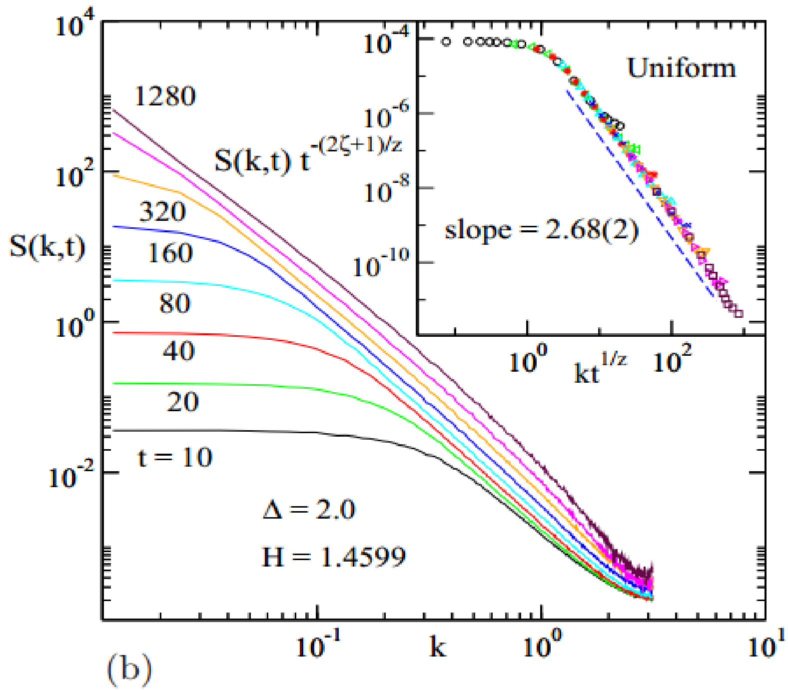

$$S(q,t) \sim q^{-(1+2\alpha)} F\left(qt^{1/z}\right)$$

$$F(x) \sim \begin{cases} x^{(1+2\alpha)} & \text{for} \quad x \ll 1 \\ 1 & \text{for} \quad x \gg 1 \end{cases}$$

depinning of the Random field Ising model

Qin, Zheng, Zhou, JPA, **45**, 115001, 2012

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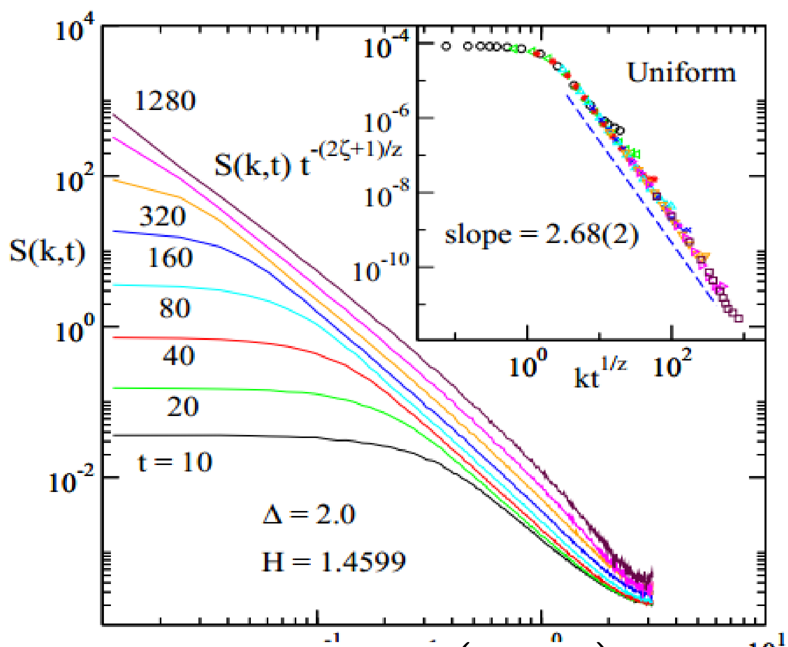
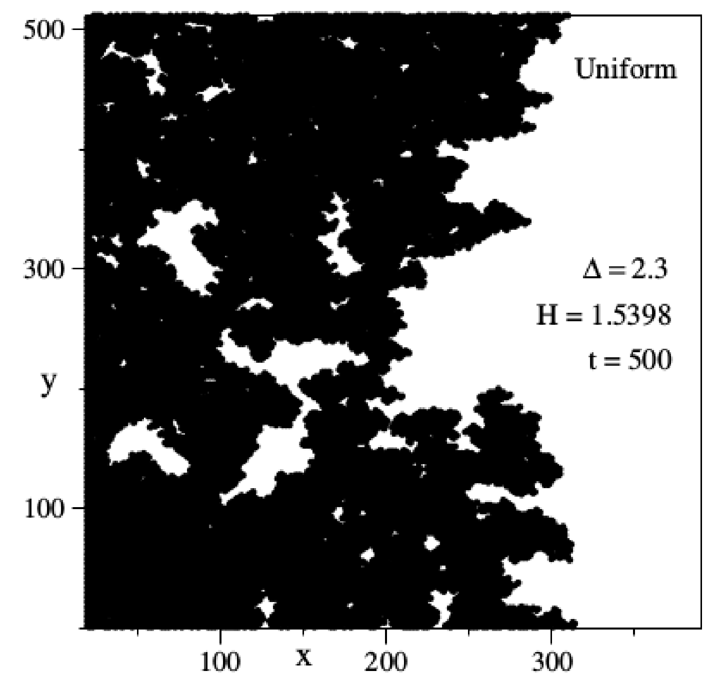
Correlation functions

structure factor

$$S(q,t) = \langle u(q,t) u(-q,t) \rangle \qquad \text{with} \qquad u(q,t) = \int dz\, u(z,t)\, e^{-iqz}$$

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - \sum_i (h_i + H) S_i,$$





$$S(q,t) \sim q^{-(1+2\alpha)} F\left(q t^{1/z}\right)$$

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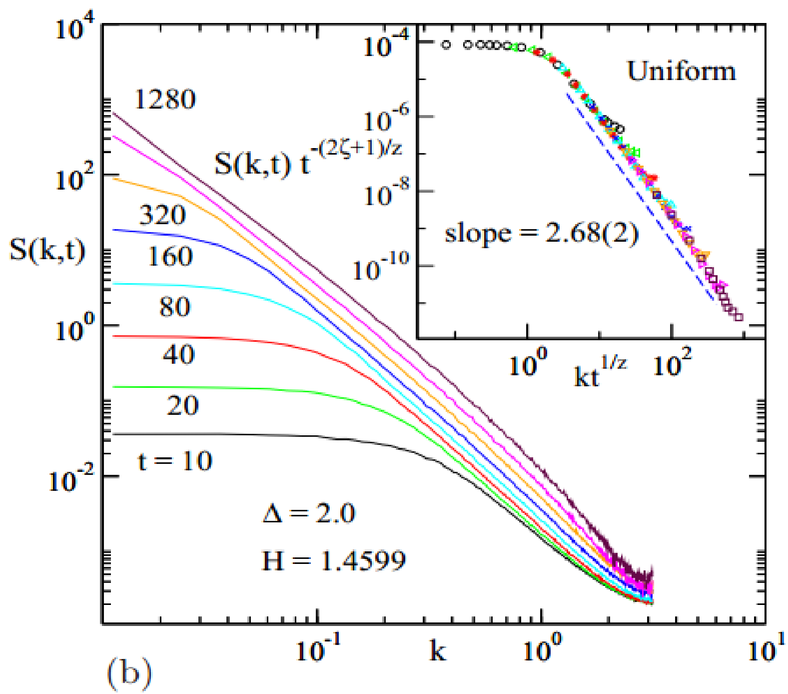

$$S(q,t) \sim t^{(1+2\alpha)/z} G\left(qt^{1/z}\right)$$

$$G(x) \sim \begin{cases} 1 & \text{for} \quad x \ll 1 \\ x^{-(1+2\alpha)} & \text{for} \quad x \gg 1 \end{cases}$$

depinning of the Random field Ising model

Qin, Zheng, Zhou, JPA, **45**, 115001, 2012

$$S(q,t) \sim \begin{cases} t^{(1+2\alpha)/z} & \text{for} \quad q \ll 1/\ell \sim t^{-1/z} \\ q^{-(1+2\alpha)} & \text{for} \quad q \gg 1/\ell \sim t^{-1/z} \end{cases}$$

Edwards-Wilkinson equation

random deposition $$\frac{\partial u(z,t)}{\partial t} = \Phi(z,t)$$

$\Phi(z,t)$: position dependent flux $$\Phi(z,t) = F + \eta(z,t)$$

F : net flux

$$\begin{aligned} \langle \eta(z,t) \rangle &= 0 \\ \langle \eta(z,t)\eta(z',t') \rangle &= D\delta(z-z')\delta(t-t') \end{aligned}$$: noise

$$\langle u(z,t) \rangle = \left\langle \int \Phi(z,t)\ dt \right\rangle = Ft + \int \langle \eta(z,t) \rangle dt$$ $$\langle u(t) \rangle = Ft$$

$$\begin{aligned} \langle u(z,t)u(z,t') \rangle &= \left\langle \int \int \Phi(z,t)\Phi(z,t')\ dt\ dt' \right\rangle \\ &= (Ft)^2 + \int \int \langle \eta(z,t)\eta(z,t') \rangle dt\ dt' \\ &= (Ft)^2 + Dt \end{aligned}$$ $$\langle u(t)^2 \rangle = (Ft)^2 + Dt$$

Edwards-Wilkinson equation

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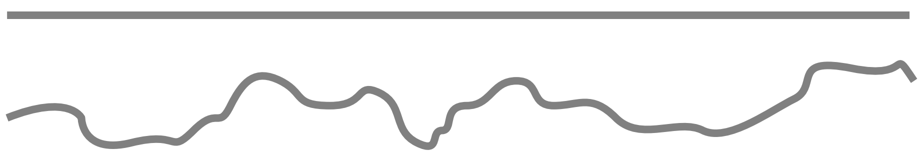
$$\langle u(t) \rangle = Ft \qquad \langle u(t)^2 \rangle = (Ft)^2 + Dt$$

$$\boxed{W^2 = Dt}$$

$$\beta = 1/2 \quad \alpha = 0$$

position and size independent – **no correlations**

Edwards-Wilkinson equation



We consider now that there is a cost in roughening the interface

Hamiltonian approach

$$\mathcal{H} = \int_{\mathcal{L}} c\,ds = c \int_{\mathcal{L}} \sqrt{dz^2 + du^2} = c \int_L \sqrt{1 + (du/dz)^2}\,dz$$

$$\begin{aligned}
\mathcal{H} &= c \int_L \sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}\,dz \\
&= c \int_L \left[1 + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{8}\left(\frac{\partial u}{\partial z}\right)^4 + \frac{1}{16}\left(\frac{\partial u}{\partial z}\right)^6 + \ldots\right] dz
\end{aligned}$$

$$\mathcal{H}_{\mathrm{el}} = \frac{c}{2} \int_L \left(\frac{\partial u}{\partial z}\right)^2 dz$$

harmonic elastic energy contribution

Edwards-Wilkinson equation

$$\gamma \frac{\partial u(z,t)}{\partial t} = -\frac{\delta \mathcal{H}[u(z,t)]}{\delta u(z,t)} + \widetilde{\eta}(z,t)$$

Langevin dynamics
- non-conserved dynaimc equation
- Model A
- overdamped equation of motion

$$\langle \widetilde{\eta}(z,t) \rangle = 0$$

$$\langle \widetilde{\eta}(z,t)\widetilde{\eta}(z',t') \rangle = 2\gamma T \delta(z-z')\delta(t-t')$$

white noise

$\frac{\delta \mathcal{H}}{\delta u}$ functional derivative

Property: **if** $\mathcal{H} = \int f\left(u, \frac{\partial u}{\partial z}\right) dz$

then $\frac{\delta \mathcal{H}}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial z}\frac{\partial f}{\partial(\partial_z u)}$

therefore: $\mathcal{H}_{\mathrm{el}} = \frac{c}{2}\int_L \left(\frac{\partial u}{\partial z}\right)^2 dz \longrightarrow \frac{\delta \mathcal{H}_{\mathrm{el}}}{\delta u} = -c\frac{\partial^2 u}{\partial z^2}$

Edwards-Wilkinson equation

$$\gamma \frac{\partial u(z,t)}{\partial t} = -\frac{\delta \mathcal{H}[u(z,t)]}{\delta u(z,t)} + \widetilde{\eta}(z,t)$$

Langevin dynamics
- non-conserved dynaimc equation
- Model A
- overdamped equation of motion

$$\langle \widetilde{\eta}(z,t) \rangle = 0$$

$$\langle \widetilde{\eta}(z,t)\widetilde{\eta}(z',t') \rangle = 2\gamma T \delta(z-z')\delta(t-t')$$

white noise

$$\frac{\partial u(z,t)}{\partial t} = \nu \frac{\partial^2 u(z,t)}{\partial z^2} + \eta(z,t)$$

Edwards-Wilkinson equation

$$\begin{aligned} \langle \eta(z,t) \rangle &= 0 \\ \langle \eta(z,t)\eta(z',t') \rangle &= \frac{2T}{\gamma}\delta(z-z')\delta(t-t') \end{aligned}$$

Edwards and Wilkinson, Proc. R. Soc.. **381**, 17, 1982

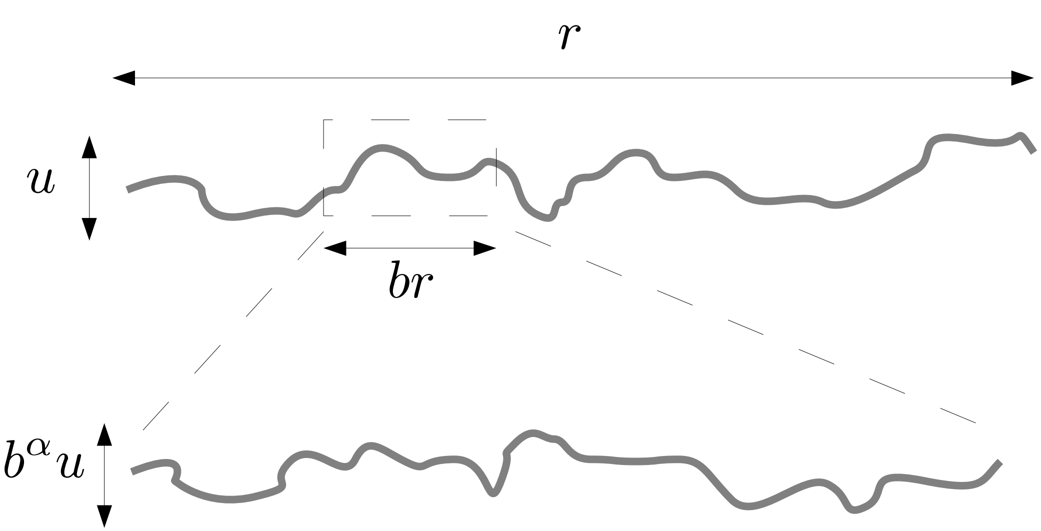
Edwards-Wilkinson equation

self-affinity: the interface is **statistically** invariant under an anisotropic transformation

$$r \to r' = br$$

$$u \to u' = b^{\alpha} u$$

$$t \to t' = b^{z} t$$



Does the EW equation contain the self-affinity propoerty?

Edwards-Wilkinson equation

$$r \to r' = br \qquad u \to u' = b^{\alpha} u \qquad t \to t' = b^{z} t$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial r^2} + \eta(z,t)$$

$$\frac{\partial (b^{\alpha} u)}{\partial (b^{z} t)} = \nu \frac{\partial^2 (b^{\alpha} u)}{\partial (br)^2} + \eta(br, b^{z} t)$$

$$b^{\alpha - z} \frac{\partial u}{\partial t} = \nu b^{\alpha - 2} \frac{\partial^2 u}{\partial r^2} + b^{-1/2 - z/2} \eta(r,t)$$

$$\frac{\partial u}{\partial t} = \nu b^{z-2} \frac{\partial^2 u}{\partial r^2} + b^{-1/2 + z/2 - \alpha} \eta(r,t)$$

$$z - 2 = 0$$

$$-1/2 + z/2 - \alpha = 0$$

$$z = 2 \quad \alpha = 1/2 \quad \beta = 1/4$$

EW exponentes

Edwards-Wilkinson equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \eta(z,t)$$

linear, partial derivatives equation
it can be solved!

Fourier representation

$$\delta u(z,t) = u(z,t) - \overline{u}(t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{iq_n z}$$

with $q_n = \frac{2\pi n}{L}$

where $\overline{u}(t) = L^{-1} \int_0^L u(z,t)\, dz$ and $c_n(t) = L^{-1} \int_0^L \delta u(z,t) e^{-iq_n z} dz$

$$\frac{\partial c_n(t)}{\partial t} = -\nu q_n^2 c_n(t) + \eta_n(t)$$

$$\langle \eta_n(t) \rangle = 0$$

$$\langle \eta_n(t) \eta_{n'}(t') \rangle = \frac{2T}{\gamma L} \delta_{n,-n'} \delta(t-t')$$

Edwards-Wilkinson equation

$$\frac{\partial c_n(t)}{\partial t} = -\nu q_n^2 c_n(t) + \eta_n(t)$$

$$\begin{aligned} \langle \eta_n(t) \rangle &= 0 \\ \langle \eta_n(t) \eta_{n'}(t') \rangle &= \frac{2T}{\gamma L} \delta_{n,-n'} \delta(t-t') \end{aligned}$$

the solution is
$$c_n(t) = c_0(0) e^{-\nu q_n^2 t} + e^{-\nu q_n^2 t} \int_0^t e^{\nu q_n^2 t'} \eta_n(t') dt'$$

The roughness can be written as
$$W^2(t) = 2 \sum_{n=1}^{\infty} \langle |c_n(t)|^2 \rangle$$

with the flat initial condition
$$c_n(0) = 0 \quad \forall n$$

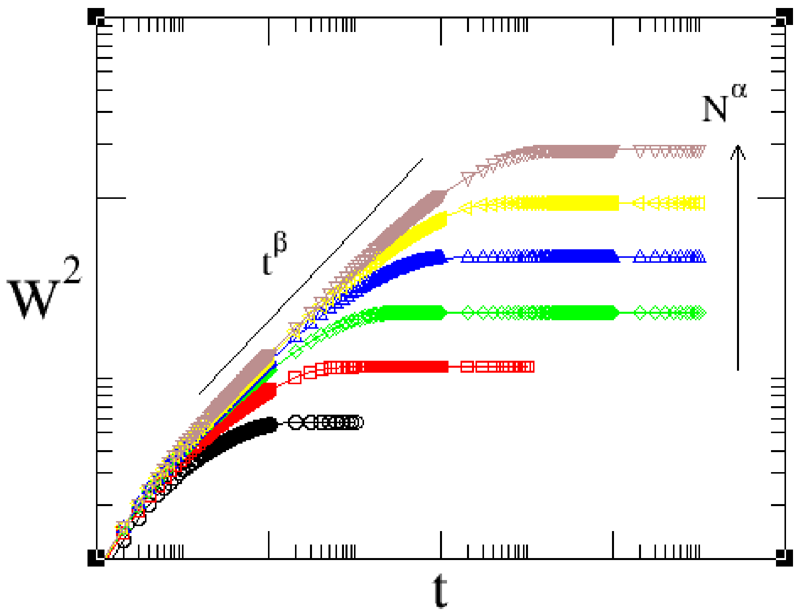
$$W^2(t) = \frac{TL}{2\pi^2 \gamma \nu} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - e^{-n^2 t / t_x} \right)$$

$$t_x = \frac{L^2}{8\pi^2 \nu}$$

growing correlation lenght

$$\xi(t) = L\sqrt{1 - e^{-t/t_x}} \sim \begin{cases} t^{1/2} & \text{for } t \ll t_x \\ L & \text{for } t \gg t_x \end{cases}$$

Edwards-Wilkinson equation



$$W^2(t) \sim \begin{cases} \frac{T}{2\gamma}\sqrt{\frac{t}{\pi\nu}} & \text{for } t \ll t_x \\ \frac{TL}{12\gamma\nu} \sim L^{2\alpha} & \text{for } t \gg t_x \end{cases}$$

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